ON THE COMPOSITION OF FIELDS

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Let K/k, K'/k be two extensions of a basic field k. By a *composite extension* of these two extensions, we understand the complex notion formed of an extension $\mathfrak{K}/k$ of k, of an isomorphism τ of K/k into $\mathfrak{K}/k$ and of an isomorphism τ' of K'/k into $\mathfrak{K}/k$, provided the following conditions are verified:

(1) $\mathfrak{K}$ is generated by the two fields K^τ, $K'^{\tau'}$.

(2) If A, A' are subsets of K, K' respectively which are algebraically independent over k, the set $A^\tau \cup (A')^{\tau'}$ is algebraically independent over k. In other words, the algebraic relations which hold in $\mathfrak{K}$ between elements of K^τ, $K'^{\tau'}$ are consequences of the algebraic relations which hold between elements of K^τ alone or of $K'^{\tau'}$ alone.[1]

THEOREM 1. *Any two given extensions K/k, K'/k have at least one composite extension.*

Let B' be a transcendence basis for K'/k. We can find a purely transcendental extension Ω/K which has a transcendence basis $B'^{\tau'}$ with the same cardinal number as B' (τ' stands for a one-to-one mapping of B' onto $B'^{\tau'}$). The algebraic closure $\bar{\Omega}$ of Ω contains the algebraic closure $\bar{\mathrm{P}}$ of the field $\mathrm{P}=k(B'^{\tau'})$. The mapping τ' may be extended to an isomorphism of K'/k with an extension $K'^{\tau'}/k$ contained in $\bar{\mathrm{P}}/k$, and a fortiori in $\bar{\Omega}$. We set $\mathfrak{K}=KK'^{\tau'}$, and denote by τ the identity mapping of K/k into $\mathfrak{K}/k$. We claim that the system $(\mathfrak{K}/k, \tau, \tau')$ is a composite extension of K/k, K'/k.

It is sufficient to check the condition (2), and we may assume without loss of generality that A, A' are finite. There exists a finite subset B_1' of B' such that $k(A', B_1')$ is algebraic over $k(B_1')$. Let d, d', e be the number of elements in A, A_1', B_1'. The elements of B_1', being algebraically independent over K, are *a fortiori* algebraically independent over $k(A)$. Therefore, the degree of transcendency of $k(A, A'^{\tau'}, B_1'^{\tau'})$ over k is $d+e$. The degree of transcendency of $k(A_1'^{\tau'}, B_1'^{\tau'})$ over $k(A_1'^{\tau'})$ is $e-d'$. The degree of transcendency f of $k(A, A'^{\tau'}, B_1'^{\tau'})$ over $k(A)$ is therefore less than or equal to $e-d$. It follows that the degree of transcendency of $k(A, A'^{\tau'})$ over k, which

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[1] The problem of composite extensions has been considered by Zariski (*Algebraic varieties over ground fields of characteristic zero*, American Journal of Mathematics, vol. 62 (1940), pp. 187–221) in the case when one of the extensions K/k, K'/k is algebraic and normal.