MEASURE AND OTHER PROPERTIES OF A HAMEL BASIS

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A Hamel basis[1] is a set $a, b, c, \cdots$ of real numbers such that if x is any real number whatsoever then x may be expressed uniquely in the form $\alpha a+\beta b+\gamma c+\cdots$ where $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. Since each of these sums is formed from a finite number of nonzero terms and the coefficients $\alpha, \beta, \gamma, \cdots$ are rational and therefore form a countable set, it seems intuitively plausible that not only should the basis set be of the same power as the continuum but in some way be of the same "thickness" as the continuum. However, this intuitive feeling is seemingly contradicted by the only known results along this line, namely: the inner measure of a Hamel basis is zero and its outer measure may also be zero.[2] Nevertheless, this intuition is justified to some extent by Theorems 2, 4, and 5. A natural question arises: In order for a set of real numbers to contain a Hamel basis, what is both necessary and sufficient? For a certain family of sets (including the Borel and analytical sets) this question is answered in two ways. Certain other properties of a Hamel basis are investigated, the most interesting being an example of a Hamel basis which contains a nonvacuous perfect set. Finally, some rather curious discontinuous solutions of the equation $f(x)+f(y)=f(x+y)$ are given.

Measure. No Hamel basis of positive exterior measure is measurable.[3] The next few theorems show this to be true also of certain transforms of every Hamel basis.

DEFINITION. *If M is a set of real numbers, by $T(M)$ is meant the set of all numbers x' such that $x'=x+(y'-y)$, where x, y, and y' belong to M.*

With M considered as a linear set, $T(M)$ is the sum of all translations of M which intersect M. For convenience, $T[T(M)]$ is abbreviated $T^2(M)$, $T[T^2(M)]$ is abbreviated $T^3(M)$, and so on and $T^0(M)=M$.

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[1] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung*: $f(x+y)=f(x)+f(y)$, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

[2] W. Sierpinski, *Sur la question de la mesurabilité de la base de M. Hamel*, Fundamenta Mathematicae, vol. 1 (1920), pp. 105–111.

[3] Ibid.