ON CONVEX SETS IN LINEAR NORMED SPACES

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M. Eidelheit has proved[1] this theorem.

THEOREM. *In a linear normed space two convex bodies (that is, convex sets with inner points) having no common inner points are separated*[2] *by a plane.*

The purpose of this note is to present a quite different and somewhat simpler proof of this result.[3]

It is known[4] for linear normed spaces that

(1) *Through every boundary point of a convex body there passes a plane supporting the body.*

A convex cone with the point x_0 as vertex is defined as a convex body C containing at least one point $x \neq x_0$ and such that for each such point x in C,

$$ax + (1 - a)x_0 \in C, \qquad a \geqq 0.$$

It is easily seen that

(2) *Every supporting plane of a convex cone C passes through the vertex x_0 of the cone.*

For, let $L(x) - b = 0$, where $L(x)$ is a linear functional and b is a constant, define a plane of support of C passing through a boundary point y of C. Suppose for definiteness that

$$L(x) - b \leqq 0$$

holds for all points x in C. Then since every point of the form $ay + (1-a)x_0$ $(a \geqq 0)$ is a boundary point of C,

$$L(ay + (1 - a)x_0) - b \leqq 0, \qquad a \geqq 0,$$

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[1] M. Eidelheit, *Zur Theorie der konvexen Mengen in linearen normierten Räumen*, Studia Mathematica, vol. 6 (1936), pp. 104–111.

[2] Two sets are separated by a plane provided they lie in opposite closed half-spaces of the plane.

[3] Added in proof: There has recently been brought to my attention another proof of Eidelheit's theorem by S. Kakutani, Proceedings of the Imperial Academy of Japan, vol. 13 (1937), pp. 93–94. The first part of the present proof is closely related to the first part of Kakutani's proof.

[4] See S. Mazur, *Über konvexen Mengen in linearen normierten Räumen*, Studia Mathematica, vol. 4 (1933), p. 74.