EXPANSIONS IN SERIES OF NON-ORTHOGONAL FUNCTIONS

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1. Introduction. Let $\phi_i(x)$ $(i=1, 2, \cdots)$ be the normalized characteristic functions of the Sturm-Liouville problem

$$\frac{d}{dx}\left(R\frac{d\phi}{dx}\right) + (\lambda P + Q)\phi = 0,$$

$$A_0\phi'(0) + B_0\phi(0) = 0, \qquad A_1\phi'(1) + B_1\phi(1) = 0,$$

in which the functions P, Q, and R are continuous, and $R>0$, $P>0$, when $0 \leqq x \leqq 1$. The set of functions $\{\phi_i(x)\}$ is closed with respect to the class $L^2(0, 1)$, in the sense that Parseval's relation,

$$\int_0^1 Pf^2dx = \sum_1^\infty \left[\int_0^1 Pf\phi_i dx\right]^2,$$

is satisfied by every function f of that class. This fact can be deduced readily from a theorem by Kellogg[1] on the completeness of the set of solutions of the self-adjoint problem of the second order. It can also be obtained from a result found by Dixon.[2]

In terms of two functions f and G of the class $L^2(0, 1)$, Parseval's relation can be written

$$\int_0^1 PfGdx = \sum_1^\infty \int_0^1 Pf\phi_i dx \int_0^1 PG\phi_i dx = \sum_1^\infty c_i \int_0^1 PG\phi_i dx,$$

where c_i are the Fourier constants of f. If $G=g/P$ when $0<x<t$ and $G\equiv 0$ when $t<x<1$, where the function g belongs to $L^2(0, 1)$, it follows that

$$\int_0^t fgdx = \sum_1^\infty c_i \int_0^t g\phi_i dx, \qquad 0 \leqq t \leqq 1. \tag{1}$$

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[1] O. D. Kellog, *Note on closure of orthogonal sets*, this Bulletin, vol. 27 (1920), pp. 165–169.

[2] A. C. Dixon, *On the series of Sturm-Liouville, as derived from a pair of fundamental integral equations instead of a differential equation*, Philosophical Transactions of the Royal Society of London, (A), vol. 211 (1912), pp. 411–432; pp. 431, 432.