AN EXTENSION OF A THEOREM OF WITT

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1. **Introduction.** If $\mathfrak{u}_1, \cdots, \mathfrak{u}_n$ is a set of vectors such that $\mathfrak{u}_i\mathfrak{u}_j = \mathfrak{u}_j\mathfrak{u}_i$ are numbers of a field K for $i, j = 1, 2, \cdots, n$, all linear combinations of these vectors with coefficients in K constitute a *vector space*

$$\mathfrak{S} = \langle \mathfrak{u}_1, \cdots, \mathfrak{u}_n \rangle$$

over K and the symmetric matrix $\mathfrak{A} = (\mathfrak{u}_i\mathfrak{u}_j) = (a_{ij})$ is the *multiplication table* for the *basis* $\mathfrak{u}_1, \cdots, \mathfrak{u}_n$. The inner product of two vectors $\sum x_i\mathfrak{u}_i$ and $\sum y_i\mathfrak{u}_i$ is the bilinear form

$$\sum (\mathfrak{u}_i\mathfrak{u}_j)x_iy_j = \sum a_{ij}x_iy_j$$

and the *norm* of a vector is the inner product of a vector and itself; it can be expressed as a quadratic form.

If $\mathfrak{C}$ is a nonsingular transformation with coefficients in K and $(\mathfrak{u}_1, \cdots, \mathfrak{u}_n)\mathfrak{C} = (\mathfrak{v}_1, \cdots, \mathfrak{v}_n)$, the $\mathfrak{v}$'s will constitute a new basis of the same space $\mathfrak{S}$ and the multiplication table for the new matrix is $\mathfrak{C}'\mathfrak{A}\mathfrak{C}$. This has the same effect on the matrix of the quadratic form $\sum a_{ij}x_ix_j$ as the transformation $(x_1, \cdots, x_n)' = \mathfrak{C}(y_1, \cdots, y_n)'$. The quadratic forms f_1 and f_2 are <u>equivalent</u> (in K) if one may be taken into the other by a nonsingular transformation with coefficients in K. Then the corresponding vector spaces are said to be <u>equivalent</u> (in K). We write $f_1 \cong f_2$ and $\mathfrak{S}_1 \cong \mathfrak{S}_2$.

It should be noted, in passing, that two vector spaces may be equivalent without being identical. For example, if $n = 3$ and

$$\mathfrak{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

it is true that $\langle \mathfrak{u}_1, \mathfrak{u}_2 \rangle \cong \langle \mathfrak{u}_2, \mathfrak{u}_3 \rangle$. However, an isomorphism may be established between two sets of vectors having the same multiplication table.

Two vectors $\mathfrak{u}$ and $\mathfrak{v}$ are *orthogonal* if $\mathfrak{u}\mathfrak{v} = 0$. Two vector spaces are orthogonal if every vector of one is orthogonal to every vector of the other. Two subspaces, $\mathfrak{S}_1$ and $\mathfrak{S}_2$, of $\mathfrak{S}$ are *complementary* if every vector of $\mathfrak{S}$ is the sum of a vector of $\mathfrak{S}_1$ and a vector of $\mathfrak{S}_2$. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are complementary orthogonal subspaces of $\mathfrak{S}$ we write

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