A CHARACTERIZATION OF THE RADICAL OF AN ALGEBRA

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1. **The first main result.** We shall prove the following result.

THEOREM 1. *Let F be any field and A an algebra over F with a unity element. Then the radical of A consists of all elements h such that $g+h$ is regular for every regular g.*

Let H be the set of all elements h defined in the theorem. It is easy to see that H is a linear set over F. We shall prove now that if A is simple, $H=0$.

Let g and g_1 be any regular elements of A and h be in H. Then $g_1^{-1}g+h$ is regular so that $g+g_1h$ is regular. Hence g_1h is in H and similarly hg_1 is in H. An arbitrary element a of A has[1] the form $a=\sum_{i=1}^{n}g_i$ with regular elements g_i so that $ah=\sum g_ih$ is a sum of elements g_ih of H. Thus ah, and similarly ha, is in H so that H is an ideal of A. If $H\neq 0$ then $H=A$ since A is simple. But A contains the regular element -1, and $(-1)+1$ is not regular so that 1 cannot be in H, whence $H\neq A$. Hence $H=0$.

Next we shall prove that $H=0$ whenever A is semi-simple. Now $A=A_1+A_2+\cdots+A_t$ where the A_i are simple, and each x of A has a unique expression $x=a_1+a_2+\cdots+a_t$ with a_i in A_i. Further, x is regular if and only if each a_i is a regular element of A_i. Let $g=g_1+\cdots+g_t$ be regular, $h=h_1+\cdots+h_t$ be in H, so that $g+h=(g_1+h_1)+\cdots+(g_t+h_t)$. Then $g+h$ is regular for every regular g if and only if g_i+h_i is regular in A_i for every regular g_i of A_i. By the proof above for simple algebras every $h_i=0$ so that $h=0$ and $H=0$.

In considering the case of a general algebra A, we show first that the radical R is contained in H. Let g be regular and r lie in R. Then $g+r$ is regular if and only if $1+g^{-1}r$ is regular. Now $g^{-1}r$ is in R, $(g^{-1}r)^t=0$ for some integer t, $(g^{-1}r)^{2t+1}+1=1$. If λ is an indeterminate, $\lambda+1$ is a factor of $\lambda^{2t+1}+1$ so that $g^{-1}r+1$ is a factor of $(g^{-1}r)^{2t+1}+1=1$; hence, $g^{-1}r+1$ is regular, $g+r$ is regular, r is in H, and R is contained in H.

It remains to prove that R contains H. Since $A-R$ is semi-simple, the set H_0 defined for $A-R$, similarly to H for A, is the zero set. If g is regular in A and h is in H, the class $[g+h]$ in $A-R$ is a regular ele-

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[1] K. Shoda, Mathematische Annalen, vol. 107 (1933), pp. 252–258.