A NOTE ON REPRESENTATION BY POLYGONAL NUMBERS

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1. **Introduction.** The universal functions of polygonal numbers of order $m+2$ were determined in an earlier paper.[1] In each universal function the number n of variables was at most $m+2$. The minimum value N of n, for each integer $m \geqq 3$, will be proved in this paper to be the integer N defined by (3). It will also be proved that there is a unique universal function having $n=N$ if and only if $m=3$, 4, $2^{N-2}-2$, $2^{N-2}-1$. A universal function having $n=N$ is given by the integers (4). At least one universal function different from this function is given by the integers (6) if $m \neq 3$, 4, $2^{N-2}-2$, $2^{N-2}-1$, and if (7) and (8) hold.

2. **Proofs.** In the notations of the paper to which reference has been made m was an arbitrary but fixed integer greater than or equal to 3. The coefficients $a_1, a_2, \cdots, a_n$ in the universal functions were positive integers to be determined, and $1 \leqq a_1 \leqq \cdots \leqq a_n$. Also, by definition, $w_k = a_1 + \cdots + a_k$ $(1 \leqq k \leqq n)$. It was proved that no function is universal if $w_n < m+2$, and that if $w_n = m+2$ then the function f is universal if and only if f is one of the following:

$$(1) \qquad (1, 1, 1, 1, 1) \text{ or } (1, 1, 1, 2), \qquad \text{with } m = 3 \text{ and } w_n = m + 2,$$

$$(2) \ (1, 1, 1, a_4, \cdots, a_n),\ w_n = m + 2 > 5,\ a_k \leqq w_{k-1} - 1\ (4 \leqq k \leqq n),$$
$$\text{but } n > 5 \text{ and } a_5 \neq 3 \text{ if } a_4 = 1.$$

Thus $N=4$ if $m=3$. It will be proved that if $m>3$ then the minimum m is the integer N uniquely defined by

$$(3) \qquad 2^{N-3} - 1 < m \leqq 2^{N-2} - 1.$$

Consider the sequence of integers $1, 1, 1, 2, 2^2, \cdots, 2^{i-3}, \cdots$ in which the ith term is 2^{i-3} if $i \geqq 3$. The sum of the first i terms is $2^{i-2}+1$ if $i \geqq 3$. Now let f be a universal function which satisfies (2). Then it is easily proved by induction that $a_k \leqq 2^{k-3}$ and $w_k \leqq 2^{k-2}+1$ $(3 \leqq k \leqq n)$. Hence if $n<N$ this would imply in particular that $m+2 = w_n \leqq 2^{n-2}+1 \leqq 2^{N-3}+1$. This contradiction of (3) when $n<N$ shows that $n \geqq N$. Furthermore $N \geqq 5$ since $m>3$ in (3).

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[1] L. W. Griffiths, *A generalization of the Fermat theorem on polygonal numbers*, Annals of Mathematics, (2), vol. 31 (1930), pp. 1–12.