THE PROBLEM OF BOLZA IN THE CALCULUS OF VARIATIONS

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1. **Introduction.** The problem of Bolza can be described briefly as the most general problem in the calculus of variations for which there exists at the present time a theory of relative maxima and minima that is comparable in completeness to those of the simpler problems in the calculus of variations. This completeness has been brought about in the last few years and it is the purpose of the present paper to discuss in some detail the results that have been achieved and the methods that have been used in obtaining them. It is impossible to give here an adequate discussion of the various aspects of the problem of Bolza. Nor is such a discussion necessary, inasmuch as three excellent reports have been given already which were concerned in whole or in part with the problem of Bolza. The first of these reports, given in 1936 by Professor Bliss [**14**],[1] is devoted to the study of the evolution of problems in the calculus of variations and, in particular, the evolution of the problem of Bolza. Professor Bliss pointed out that even Euler and Lagrange formulated problems that are of essentially the same generality as the problem of Bolza. Moreover, they derived in a formal way the Euler-Lagrange equations that the solutions of the problems must satisfy. The second report was given in 1937 by Professor Reid [**17**] and was concerned with boundary value problems and their relations to problems in the calculus of variations. In particular, he pointed out the various relationships between the problem of Bolza and boundary value problems. The third and final report was given in 1938 by Professor McShane [**22**] who outlined the progress that had been made in the calculus of variations during the preceding twenty-five years. Here the essential achievements in the theory of the problem of Bolza were described but nothing was said as to how these results were obtained. It is the purpose of the present paper to discuss certain interesting and important aspects of the theory of the problem of Bolza that Professors Bliss, Reid and McShane of necessity had to omit or describe inadequately in their reports. In particular I shall describe to you the basic ideas involved in the sufficiency proofs for relative minima. The progress made in

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[1] Numbers in brackets refer to the list of papers at the end of this paper.