THE GEOMETRY OF WHIRLS AND WHIRL-MOTIONS IN SPACE[1]

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1. **Introduction.** The geometry of *whirls* and *whirl-motions* in the plane was inaugurated by E. Kasner.[2] An adaptation of Kasner's geometry to the sphere was made by K. Strubecker.[3] It is the purpose of this paper to develop a strictly analogous geometry in euclidean three-space, S_3. To render such a development possible we shall introduce a new type of *oriented* plane element—namely, a geometric object formed by a plane, a point in the plane, and an ordered pair of orthogonal fundamental directions in the plane; the fundamental directions shall be given by a pair of unit vectors in the plane. There are $2\infty^6$ such plane elements in S_3; there are ∞^5 of the Lie kind. Henceforth, *plane element* shall mean only the new kind of plane element.[4]

2. **Turns, slides, and direct whirls.** Let e_0, e_1, e_2, e_3 be the Hamiltonian quaternion units such that

$$e_0 e_i = e_i e_0 = e_i, \qquad e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1.$$

Let any real point P in S_3 whose orthogonal cartesian coordinates are z_1, z_2, z_3 be represented by the position vector $z = z_1 e_1 + z_2 e_2 + z_3 e_3$. Let

[1] Presented to the Society February, 22, 1941.

[2] Edward Kasner, *The group of turns and slides and the geometry of turbines*, American Journal of Mathematics, vol. 33 (1911), pp. 193–202. Further development of the subject appears in a series of papers by Kasner and De Cicco; see, for example, their joint papers, *Quadratic fields in the geometry of the whirl-motion group* G_6, ibid., vol. 61 (1939), pp. 131–142; and *The geometry of the whirl-motion group* G_6: *elementary invariants*, this Bulletin, vol. 44 (1938), pp. 399–403.

[3] K. Strubecker, *Zur Geometrie sphärischer Kurvenscharen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 44 (1934), pp. 184–198.

[4] Inasmuch as our *oriented plane element* defines a position of a rigid body in space, it is essentially equivalent to Study's *soma*, *Geometrie der Dynamen*, Leipizg, 1903, and to De Saussure's *feuillet*, *Exposé Résumé de la Géométrie des Feuillets*, Geneva, 1910; see also R. Bricard, Nouvelles Annales de Mathématiques, (4), vol. 10 (1910). It was remarked by Kasner in his 1911 paper, loc. cit., that it was possible to obtain in any space of constant curvature a group analogous to his group of whirls in the plane and that, moreover, for ordinary space the *feuillet*, consisting of a point, line, and plane all incident with one another, would be an appropriate element. Another generalization of Kasner's turbine geometry, along lines different from those pursued in this paper, has been carried out by A. Narasinga Rao, *Studies in turbine geometry* I, Journal of the Indian Mathematical Society, vol. 3 (1938), pp. 96–108; II, Proceedings of the Indian Academy of Sciences, vol. 8A (1938), pp. 179–186.