ZERO-DIMENSIONAL FAMILIES OF SETS[1]

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A family $\Phi=\{A_\alpha\}$ of subsets of a topological space X will be called 0-*dimensional* if given an open set U such that $A_{\alpha_0}\subset U$, there is an open set V such that (1) $A_{\alpha_0}\subset V\subset U$ and (2) $(\overline{V}-V)\sum_\alpha A_\alpha=0$. We enumerate below a few of the most common 0-dimensional families. In each case the proof of 0-dimensionality is easy, and is therefore omitted.

(I) Every family of disjoint open subsets of a topological space is 0-dimensional.

(II) Let Y be a locally connected subset of a topological space X. The family Φ of the components of Y is 0-dimensional.

(III) Let Y be a compact and closed subset of a metric space X. The family Φ of the components of Y is 0-dimensional.

(IV) Let Y be a subset of a metric space X. The family Φ consisting of the individual points of Y is 0-dimensional if and only if $\dim Y=0$.

(V) Let Φ be a family of closed subsets of a compact metric space X. If, given any sequence $F, F_1, F_2, \cdots$ of sets of Φ, the relation $F\cdot\liminf F_i\neq 0$ implies $\liminf F_i\subset F$, then the family Φ is called *upper-semi-continuous*. In this case the sets of the family Φ are disjoint. There is a standard way of introducing a topology into the family Φ which leads to a separable metrizable *hyperspace* Φ^*. The family Φ is 0-dimensional if and only if $\dim \Phi^*=0$. In particular, Φ is 0-dimensional whenever it is upper-semi-continuous and countable.

(VI) Let Y be a subset of a topological space X and let Y be homeomorphic with a subset of the linear continuum. The family Φ of the components of Y is 0-dimensional.

The purpose of this note is to establish the following theorem:

THEOREM. *Let X be a unicoherent Peano continuum,[2] $\Phi=\{A_\alpha\}$ a 0-dimensional family of subsets of X, and x_1 and x_2 two points of X. If none of the sets A_α cuts X between x_1 and x_2,[3] then $\sum_\alpha A_\alpha$ does not cut X between x_1 and x_2.*

Various corollaries can be obtained by taking X to be the n-sphere

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[1] Presented to the Society, December 26, 1939, under the title *On 0-dimensional upper-semi-continuous collections.*

[2] A Peano continuum X is called *unicoherent* if given any decomposition $X=X_1+X_2$ into continua, the set $X_1\cdot X_2$ is a continuum.

[3] A set $A\subset X$ *cuts* X between x_1 and x_2 if $X-A$ contains no continuum joining x_1 and x_2.