CESÀRO SUMMABILITY OF A CLASS OF SERIES

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The following theorem has recently been proved by H. L. Garabedian:[1] If a_n is a polynomial in n of degree $k-1$, the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is summable (C, k) but not $(C, k-1)$ to the value $\sum_{i=0}^{k-1} 2^{-i-1}\Delta^i a_0$. Here and elsewhere in this paper k is understood to be a fixed positive integer.

Our present object is to obtain some extensions of this result. One of these may be stated at once, the proof being given at the end of the paper.

THEOREM 2. *Let a_n be a polynomial in n of degree $k-1$, and let z be a complex number such that $|z|=1$, $z\neq 1$. Then the series $\sum_{n=0}^{\infty} a_n z^n$ is summable (C, k) but not $(C, k-1)$ to the value $-\sum_{m=0}^{k-1} z^m (z-1)^{-m-1}\Delta^m a_0$.*

Before stating our first theorem we require the following definitions. Let f_n be a periodic function of the integer n; that is to say, let there be an integer p such that $f_{n+p}=f_n$ for all values of n. Let $M(f)$ denote the mean value of f_n over a period, thus $M(f)=(1/p)\sum_{n=0}^{p-1} f_n$. Suppose now that f_n is a periodic function of mean value zero; that is, let $M(f)=0$. Writing $f(n)$, $a(n)$ in place of f_n, a_n, set

$$\text{(a)}\qquad \begin{aligned} f_1(n) &= \sum_{j=0}^{n} f(j), & c_1 &= M(f_1), & F_1(n) &= f_1(n) - c_1,\\ f_2(n) &= \sum_{j=0}^{n} F_1(j), & c_2 &= M(f_2), & F_2(n) &= f_2(n) - c_2,\\ f_3(n) &= \sum_{j=0}^{n} F_2(j), & c_3 &= M(f_3), & F_3(n) &= f_3(n) - c_3, \end{aligned}$$

and so on; this procedure ensures that $F_i(n)$ is periodic with mean value zero $(i=1, 2, \cdots)$. In terms of these definitions we prove the following theorem:

THEOREM 1. *Let $a(n)$ be a polynomial in n of degree $k-1$, and $f(n)$ a periodic function with mean value zero. Then the series $\sum_{n=0}^{\infty} f(n)a(n)$ is summable (C, k) but not $(C, k-1)$ to the value $\sum_{i=0}^{k-1} c_{i+1}\Delta^i a_0$.*

If $f(n)=(-1)^n$ it is easily verified that $c_{i+1}=2^{-i-1}$, which shows

[1] This Bulletin, vol. 45 (1939), pp. 592–596. In the theorem as here stated it is required that a_n satisfy the conditions $\Delta^{k-1}a_0\neq 0$, $\Delta^i a_0 = 0$ $(i\geqq k)$. It is easily seen that this is equivalent to requiring a_n to be a polynomial in n of degree $k-1$.