THE FINITE DIFFERENCES OF POLYGENIC FUNCTIONS[1]

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By a polygenic function $f(z)$ we shall mean a function analytic in x and y separately, but whose real and imaginary parts are not required to satisfy the Cauchy-Riemann equations. At any point z the derivative of such a function will depend on θ, the angle at which the incremented point (used in defining the derivative) approaches z. The set of these numbers, for a fixed z, but for different θ, form a circle. The equation for the derivative was given by Riemann in his classic dissertation (1851), but Kasner was the first to point out that it was a circle and make a detailed study of its geometry.[2] Hedrick called it the Kasner circle.

In this paper we shall be concerned with the finite difference quotients of polygenic functions. We shall show how a surface can be constructed for each point z representing the difference quotient, and the derivative circle is a cross section of this surface.

The conjugate form. Regard

$$z = x + iy, \qquad \bar{z} = x - iy$$

as a linear substitution, and perform its inverse

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z})$$

on $f(z)$. The resulting $F(z, \bar{z})$ will be called the conjugate form of f. Let $D_z F$ and $D_{\bar{z}} F$ be the partial derivatives[3] of $F(z, \bar{z})$, regarding z and $\bar{z}$ as independent variables. That is,

$$(1) \qquad \begin{aligned} D_z F &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \tfrac{1}{2}(D_x - D_y)f, \\ D_{\bar{z}} F &= \tfrac{1}{2}(D_x + D_y)f. \end{aligned}$$

The operator E^{ω}. Let $\omega = \rho e^{i\theta}$. We define

$$E^{\omega} f(z) = f(z + \omega).$$

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[2] *General theory of polygenic or non-monogenic functions*; *The derivative congruence of circles*, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82. *A new theory of polygenic functions*, Science, vol. 66 (1927). Also, *The Geometry of Polygenic Functions*, Kasner and DeCicco—a book in the course of preparation.

[3] In Kasner's notation, these are $\mathfrak{M}(f)$ and $\mathfrak{P}(f)$.