POWER SERIES THE ROOTS OF WHOSE PARTIAL SUMS LIE IN A SECTOR[1]

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If the roots of the partial sums of a power series $f(z)=\sum a_n z^n$ lie in a sector with vertex at the origin and aperture $\alpha<2\pi$, the power series cannot have a positive finite radius of convergence.[2] But if $f(z)$ is an entire function, the roots of its partial sums may lie in such a sector. The question arises: what restrictions are imposed on $f(z)$ by the requirement that α be sufficiently small, say $\alpha<\pi$? According to a theorem of Pólya the order of $f(z)$ must be not greater than 1 if the radius of convergence of the power series is positive.[3] Without this assumption the investigation which follows shows that *if $\alpha<\pi$, $f(z)$ is an entire function of order* 0. This result was obtained by Pólya for the case in which $\alpha=0$.[4]

LEMMA. *If the complex numbers $z_1, \cdots, z_n$ $(z_1 \cdots z_n \neq 0)$ lie in a sector with vertex at the origin and aperture $\alpha<\pi$, then*

$$\frac{n \cos \alpha/2}{\left|\sum_{k=1}^{n} z_k^{-1}\right|} \leqq \left| z_1 \cdots z_n \right|^{1/n} \leqq \frac{1}{n} \sec \alpha/2 \left| \sum_{k=1}^{n} z_k \right|. \tag{1}$$

When $\alpha=0$ equality occurs if and only if $z_1=\cdots=z_n$. When $\alpha>0$ equality occurs if and only if n is even and $n/2$ of the numbers are equal to $re^{i\phi}$ $(r>0;\ 0\leqq\phi<2\pi)$ and the other $n/2$ numbers are equal to $re^{i(\phi+\alpha)}$.

Suppose first that the sector is $-\alpha/2\leqq \text{am } z\leqq\alpha/2$. Let the n numbers be

$$z_k = \left| z_k \right| e^{i\theta_k}, \qquad -\alpha/2 \leqq \theta_k \leqq \alpha/2;\ k=1,\cdots,n.$$

Since

$$\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} \left| z_k \right| \cos \theta_k + i \sum_{k=1}^{n} \left| z_k \right| \sin \theta_k, \tag{2}$$

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[2] This follows from Jentzsch's theorem: every point on the circle of convergence of a power series is a limit point of roots of its partial sums. See R. Jentzsch, *Untersuchungen zur Theorie der Folgen analytischer Funktionen*, Acta Mathematica, vol. 41 (1917), p. 219; E. C. Titchmarsh, *Theory of Functions*, 1932, p. 238.

[3] G. Pólya, *Ueber Annäherung durch Polynome deren sämtliche Wurzeln in einen Winkelraum fallen*, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, 1913, pp. 325–330.

[4] G. Pólya, *Ueber Annäherung durch Polynome mit lauter reellen Wurzeln*, Rendiconti del Circolo Matematico di Palermo, vol. 36 (1913), pp. 279–295.