TRANSLATED FUNCTIONS AND STATISTICAL INDEPENDENCE[1]

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1. **Introduction.** Professor Steinhaus[2] proposed the question whether there exists a function $f(t)$ defined in $(-\infty, \infty)$ such that, for each sequence $\lambda_1, \lambda_2, \cdots$ of different real numbers, the "translated" functions $f(t+\lambda_1), f(t+\lambda_2), \cdots$ form a sequence of statistically independent functions.[3] We shall answer this question in the affirmative by giving concrete examples, and shall discuss some related problems.

2. **Notation and lemmas.** Let

$$\mathfrak{M}\{f(t)\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt \tag{1}$$

denote as usual the *mean value* of $f(t)$ in case the limit exists. If E is a point set on the line $(-\infty, \infty)$ and $g(t)$ is the characteristic function of E, the mean value $\mathfrak{M}\{g(t)\}$ (if it exists) is called the *relative measure* of E and will be denoted by $|E|$. A non-decreasing function $\sigma(\alpha)$ such that $\sigma(-\infty)=0$, $\sigma(+\infty)=1$ is called an *asymptotic distribution function* of $f(t)$ if

$$\left| \underset{t}{E}\{f(t) < \alpha\} \right| = \sigma(\alpha) \tag{2}$$

at each point of continuity of $\sigma(\alpha)$. A set $f_1(t), \cdots, f_n(t)$ of functions having asymptotic distribution functions $\sigma_1(\alpha), \cdots, \sigma_n(\alpha)$ is called *statistically independent* if

$$\left| \underset{t}{E}\{f_1(t) < \alpha_1; \cdots ; f_n(t) < \alpha_n\} \right| = \sigma_1(\alpha_1) \cdots \sigma_n(\alpha_n) \tag{3}$$

for each set $\alpha_1, \cdots, \alpha_n$ of real numbers such that α_k is a point of continuity of $\sigma_k(\alpha)$. An infinite set of functions is called statistically independent if each finite subset is statistically independent.

Our proofs will be based on the following theorem of Kac and Steinhaus, loc. cit.

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[1] Presented to the Society, February 24, 1940.

[2] In a letter to one of the authors. The problem arose in connection with the theory of turbulence, but because of the outbreak of the war we have not been able to get details.

[3] See M. Kac and H. Steinhaus, *Sur les fonctions indépendantes* IV, Studia Mathematica, vol. 7 (1937), pp. 1–15; and P. Hartman, E. R. van Kampen, and A. Wintner, *Asymptotic distributions and statistical independence*, American Journal of Mathematics, vol. 61 (1939), pp. 477–486.