NON-CYCLIC ALGEBRAS OF DEGREE FOUR AND EXPONENT TWO WITH PURE MAXIMAL SUBFIELDS[1]

ROY DUBISCH

In a recent paper[2] A. A. Albert proved the falsity of the converse of the well known proposition that a cyclic normal division algebra contains a quantity j whose minimum equation is $x^n = j$ in the base field of the algebra. The proof consists in giving an example of a non-cyclic normal division algebra containing a quantity j as described above. The algebra described in that example was of degree and exponent four. It is the purpose of this paper to show that the exponent does not affect the property, and this we shall do by constructing similarly an algebra of degree four whose exponent is two.

We shall actually prove the following theorem.

THEOREM. *Let ξ and η be independent indeterminants over the field R of real numbers, $K = R(\xi, \eta)$. Then there exist non-cyclic normal division algebras of degree four and exponent two over K such that $t^4 = \gamma$ in K, t^2 not in K, for some quantity t in each algebra A.*

To make our proof we use the known[3] property stating that a normal division algebra of degree four has exponent two if and only if it is expressible as a direct product of two algebras of degree two. Therefore we may take our desired algebra A to be

$$\begin{aligned} &A = B \times C = (1, i, j, ij) \times (1, x, y, xy), \\ (1)\qquad &ji = -ij, \quad i^2 = u, \quad j^2 = a, \qquad u \neq 0,\ a \neq 0 \text{ in } K, \\ &yx = -xy, \quad x^2 = v, \quad y^2 = b, \qquad v \neq 0,\ b \neq 0 \text{ in } K. \end{aligned}$$

We seek first a quantity t with minimum equation $t^4 = \gamma$ in K. Now if we take $t = a_1 i + a_2 j + a_3 ij$, where a_1, a_2, a_3 are in $K(x)$, and if we put

$$(2)\qquad a_1 = c_1 + c_2 x, \quad a_2 = d_1 + d_2 x, \quad a_3 = f_1 + f_2 x, \qquad c_i, d_i, f_i \text{ in } K,$$

we can easily compute that if

$$(3)\qquad a = -\frac{c_1^2 u + c_2^2 uv}{d_1^2 + d_2^2 v - f_1^2 u - f_2^2 uv},$$

[1] Presented to the Society, April 13, 1940.

[2] A. A. Albert, *Non-cyclic algebras with pure maximal subfields*, this Bulletin, vol. 44 (1938), pp. 576–579.

[3] A. A. Albert, *Normal division algebras of degree four*, Transactions of this Society, vol. 34 (1932), p. 369.