A NOTE ON A THEOREM OF RADÓ CONCERNING THE $(1, m)$ CONFORMAL MAPS OF A MULTIPLY-CONNECTED REGION INTO ITSELF[1]

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Let G_z denote a region in the z-plane and let $w=f(z)$ be a function of z defined for $z\subset G_z$ which has the following properties: (1) $w=f(z)$ is analytic and single-valued for $z\subset G_z$, (2) $z\subset G_z$ implies that $f(z)\subset G_z$, (3) to each point $w_0\subset G_z$ there correspond m and only m points $z_0^{(k)}$ $(k=1, 2, \cdots, m)$ contained in G_z such that $f(z_0^{(k)})=w_0$ $(k=1, 2, \cdots, m)$ where following the usual convention we count the $z_0^{(k)}$ according to their multiplicities. Then $w=f(z)$ is said to define a $(1, m)$ *conformal map of G_z onto itself*. Such maps have been studied by Fatou[2] and Julia[3] for the case where G_z is simply-connected, and by Radó[4] who treated multiply-connected regions as well. Among the results which Radó established is the following theorem:

Let G_z be a region of finite connectivity p (>1); then there exists no $(1, m)$ conformal map of G_z onto itself for $m>1$.

Let us remark with Radó that the theorem is no longer valid if G_z is of infinite connectivity, as simple examples from the theory of the iteration of rational functions show.[5] Radó's proof of the theorem just cited is based on the possibility of mapping one-to-one and conformally a region of finite connectivity p, none of the components of its boundary reducing to points, onto a region of connectivity p, the boundary of which consists of p disjoint circles. Other types of canonical regions yield the same result, notably one due to Koebe.[6] It is the object of the present note to establish Radó's theorem directly without appeal to the possibility of mapping one-to-one and conformally the region G_z onto a canonical region. Our tools are the theory of iteration and a simple modification of Nevanlinna's principle of harmonic measure.[7]

Let G_z, the region we are going to study, have as its boundary p (>1) disjoint continua Γ_k $(k=1, 2, \cdots, p)$. It is evident that we

[1] Presented to the Society, April 27, 1940, under the title *A note on a theorem of Radó*.

[2] P. Fatou, Bulletin de la Société Mathématique de France, 1919.

[3] G. Julia, Comptes Rendus de l'Académie des Sciences, Paris, vol. 166 (1918).

[4] T. Radó, Acta Szeged, vol. 1 (1922).

[5] G. Julia, Journal de Mathématiques Pures et Appliquées, 1918.

[6] P. Koebe, Acta Mathematica, vol. 41.

[7] R. Nevanlinna, *Eindeutige analytische Funktionen*, chap. 3.