LINEAR FORMS IN FUNCTION FIELDS[1]

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We shall prove algebraically an analogue for function fields[2] of a well known theorem of Minkowski on linear forms.[3]

THEOREM 1. *Let F be a field and z an indeterminate over F. Let*

$$L_i = \sum_{j=1}^{n} a_{ij}x_j, \qquad i = 1, \cdots, n, \tag{1}$$

be n linear expressions with coefficients a_{ij} in $F(z)$ and with the determinant $|a_{ij}|$ of degree[4] *d. Then for any set of n integers $c_1, \cdots, c_n$ which satisfy the condition $\sum_{i=1}^{n} c_i > d-n$ there exists a set of values for $x_1, \cdots, x_n$ in $F[z]$ and not all zero such that each L_i has degree at most c_i.*

First, we may assume that all of the c_i are equal. For, suppose that c is the maximum of the c_i. Write L_i' for $L_i z^{c-c_i}$. The determinant of the coefficients of the L_i' has degree $d' = d + \sum (c - c_i) < \sum c + n$. If there is a set of values for $x_1, \cdots, x_n$ with the property that the degree of each L_i' is at most c, then these same values will make the degree of L_i at most c_i.

Next, we may assume, after multiplying each L_i by a suitable polynomial and by using an argument similar to that above, that all the a_{ij} are in $F[z]$.

We shall now convert our system of L_i by means of a transformation of determinant unity with elements in $F[z]$ into an equivalent system having $a_{ij}=0$ for $i<j$. Let b_1 be the g.c.d. of the a_{1j}; then $b_1 = \sum_{j=1}^{n} a_{1j}c_{j1}$ for appropriate c_{j1} in $F[z]$. Necessarily the c_{j1} are relatively prime. It is possible to find other quantities c_{jk} $(k=2, \cdots, n)$ such that the determinant $|c_{jk}|$ has value unity.[5] Thus the transfor-

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[2] See M. Deuring, *Zur Theorie der Idealklassen in algebraischen Funktionenkörpern*, Mathematische Annalen, vol. 106 (1932), pp. 103–106, for a related result. I believe the results I prove are new.

[3] A bibliography of both analytic and algebraic proofs of the theorem of Minkowski on linear forms is given by E. Jacobsthal, *Der Minkowskische Linearformensatz*, Sitzungsberichte Berliner mathematischen Gesellschaft, vol. 33 (1934), pp. 62–64. See also L. J. Mordell, *Minkowski's theorem on homogeneous linear forms*, Journal of the London Mathematical Society, vol. 8 (1933), pp. 179–192.

[4] The degree of a rational function is the degree of the numerator less that of the denominator. Zero is assigned the degree minus infinity.

[5] A. A. Albert, *Normalized integral bases of algebraic number fields* I, Annals of Mathematics, (2), vol. 38 (1937), p. 926 ff. The statement is proved for rational integral c_{jk} but the proof applies to any integral domain having the property that a