AN UNSYMMETRIC FUBINI THEOREM[1]

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Because of the great variety of ways in which repeated Stieltjes integrals may occur, the ordinary Fubini theorem for Lebesgue integrals has more than one analogue in Lebesgue-Stieltjes (Radon) integrals.[2] One of these (the symmetric one) is well known.[3] However, there is another form which is not symmetric, and its proof does not seem to be in the literature, although it is known to many writers, in special cases at least.[4] This form does not appear to be immediately derivable from the symmetric form, and since it is of interest in various connections, it seems worthwhile to give an explicit statement and proof. We do not follow the usual procedure of beginning with finite limits and then allowing the limits to become infinite because the passage to infinity seems to present difficulties of the same order as the direct proof of the final result itself. The immediate use of infinite limits is made possible by the fact that the symmetric Fubini theorem has already been proved with infinite limits.

THEOREM. *Let $k(x)$ be a function of bounded variation on every finite interval. Let $p(x, u)$ be Borel measurable in (x, u); for almost all x with respect to $k(x)$ let it be of bounded variation in u over every finite u-interval. Denote by $V(x, u)$ the variation*[5] *$V(x, u)=\int_{0+0}^{u+0}|d_v p(x, v)|$. Assume that $\int_{-\infty}^{\infty} V(x, u)|dk(x)|$ exists (is finite) for all*[6] *u. Let $s(u)$ be Borel measurable on $(-\infty, \infty)$. Then the existence (finiteness) of either*

$$\int_{-\infty}^{\infty} | s(u) | \, d_u \int_{-\infty}^{\infty} V(x, u) \, | dk(x) | \tag{1}$$

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[2] The integrals which occur in this paper are understood to be Lebesgue-Stieltjes (Radon) integrals. See, for example, Saks, *Theory of the Integral*, 2d revised edition, Warsaw-Lemberg, 1937, pp. 19 and 67. We note that the familiar principles of monotone and dominated convergence in the Lebesgue theory are also valid in this theory; see pages 28 and 29. In this paper we shall not admit $\pm\infty$ as members of our number system; that is, existence implies finiteness.

[3] Saks, loc. cit., p. 81.

[4] See for instance, N. Wiener and H. R. Pitt, *On absolutely convergent Fourier-Stieltjes transforms*, Duke Mathematical Journal, vol. 4 (1938), pp. 420–436. The reader will note that a form of this theorem was used in passing from line 13 to line 14 on page 421.

[5] For negative u we understand that $V(x, u)=-\int_{u+0}^{0+0}|d_v p(x, v)|$. Of course V is strictly a variation only for positive u.

[6] Clearly it would be sufficient to require that the integral be finite for u ranging over some sequence having $\pm\infty$ as limit points.