ON THE CONVEX SOLUTION OF A CERTAIN FUNCTIONAL EQUATION[1]

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The object of this paper is to give a generalization of a result obtained in a recent article by Mayer.[2] His result is a particular case (for $a=1$, $p=1$, $K=0$) of the following theorem.

THEOREM 1. *The only function which is convex for all* $x>K\geqq 0$, *and satisfies the functional equation*

$$1/f(x+a) = x^p f(x), \qquad x>0,\ a>0,\ p>0, \tag{1}$$

is

$$F(x) = \left[\frac{\Gamma(x/2a)}{(2a)^{1/2}\Gamma((x+a)/2a)}\right]^p. \tag{2}$$

The following proof follows closely that given by Mayer in the paper to which we referred above. All real functions occurring in this paper will be assumed to be defined for all positive values of the independent variable unless otherwise specified. A function $f(x)$ is said to be convex for all $x>K$ if for every pair of values of x, say x_1 and x_2, for which $f(x)$ is defined, and $x_1>K$, $x_2>K$,

$$f\left(\frac{x_1+x_2}{2}\right) \leqq \tfrac{1}{2}[f(x_1)+f(x_2)]. \tag{3}$$

That $f(x)$ satisfies the functional equation (1) follows from the fundamental equation for the Gamma function, $\Gamma(x+1)=x\Gamma(x)$.

We shall now show that $F(x)$ is convex. From (2) it follows[3] that

$$\frac{F'(x)}{F(x)} = -\,ap\sum_{v=0}^{\infty}\frac{1}{(2av+x)(2av+a+x)}\,.$$

Since $a>0$, $p>0$, it follows that $F'(x)/F(x)$ is increasing with increasing x. Log $F(x)$ is therefore a convex function, and therefore $F(x)$, for the convexity of a function follows from the convexity of its logarithm as can be easily verified.

Since $F'(x)<0$, $F(x)$ is a decreasing function of x, and we see by (1) that

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[1] Presented to the Society, April 13, 1940.

[2] A. E. Mayer, Acta Mathematica, vol. 70 (1938), pp. 57–62.

[3] Mayer, loc. cit.