NOTE ON CERTAIN ORTHOGONAL POLYNOMIALS[1]

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1. **Introduction.** A well known theorem[2] states that if $K_n(x, t) = \sum_0^n p_k(x)p_k(t)$ is the kernel associated with a system of orthonormal polynomials on an interval (a, b) with weight function $\rho(t)$, and if x_0 is a real number not belonging to the open interval (a, b), the function $K_n(x, t)$ is orthogonal to every polynomial of lower degree with respect to $|t-x_0|\rho(t)$ as weight function. This result is to be extended below to orthogonal trigonometric sums, and more generally to other orthogonal polynomials in two real variables on an algebraic curve.[3] The fact that a single polynomial of the nth degree in the original formulation is replaced by two or more sums or polynomials of like degree in the generalized orthogonal systems imparts to the extension some features of novelty. Certain other generalizations are briefly mentioned also.

2. **Trigonometric sums.** Let $U_0(x)$, $U_1(x)$, $V_1(x)$, $\cdots$ be orthonormal trigonometric sums for weight $\rho(x)$, and let

$$\begin{aligned} K_n(x, s) = U_0(x)U_0(s) + U_1(x)U_1(s) + \cdots + U_n(x)U_n(s) \\ + V_1(x)V_1(s) + \cdots + V_n(x)V_n(s). \end{aligned}$$

For definiteness it may be assumed that $U_k(x)$ contains no term in $\sin kx$, while $V_k(x)$ contains $\sin kx$ with a nonvanishing coefficient; the function $K_n(x, s)$ would be unchanged if U_k, V_k were replaced by any equivalent pair of sums of the kth order. If $T_n(x)$ is any trigonometric sum of the nth order,

$$\int_{-\pi}^{\pi} \rho(s)K_n(x, s)T_n(s)ds = T_n(x).$$

Suppose $\rho(x) \equiv 0$ *throughout an interval* (α, β). Let x_1, x_2 be any two distinct points of (α, β). Let $\tau_{n-1}(x)$ be an arbitrary trigonometric sum of order $n-1$ at most, and let $\sin \frac{1}{2}(x-x_1) \sin \frac{1}{2}(x-x_2)\tau_{n-1}(x) = T_n(x)$; the product

$$\sin \tfrac{1}{2}(x - x_1) \sin \tfrac{1}{2}(x - x_2) = \tfrac{1}{2} \cos \tfrac{1}{2}(x_1 - x_2) - \tfrac{1}{2} \cos \left[x - \tfrac{1}{2}(x_1 + x_2)\right]$$

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[2] See, e.g., G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, New York, 1939, p. 39.

[3] See D. Jackson, *Orthogonal polynomials on plane curves*, Duke Mathematical Journal, vol. 3 (1937), pp. 228–236.