FOURIER COEFFICIENTS OF BOUNDED FUNCTIONS[1]

BERNARD FRIEDMAN

The results of this paper are divided into two parts. First: inequalities for the Fourier coefficients of any bounded function; second: an approximation theorem for the Fourier development of an arbitrary bounded function.

Inequalities for Fourier coefficients have been discussed in a paper by Professor Szász.[2] However, his work deals mainly with linear inequalities for complex coefficients. The inequalities to be investigated in this paper are not linear. Nevertheless, they are the best possible, for this reason: given any set of numbers which makes the inequality an equality, there exists a bounded function which has these numbers as its Fourier coefficients.

A simple illustration will clarify this. Let $f(x)$ be a bounded measurable function in $(-\pi, \pi)$ such that $|f(x)| \leqq 1$. The Fourier coefficients of $f(x)$ are given by the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx,$$

$$n = 1, 2, 3, \cdots .$$

Then, it is clear that

$$|a_n| \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|\,|\cos nx|\, dx \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos nx|\, dx,$$

$$|b_n| \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|\,|\sin nx|\, dx \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin nx|\, dx$$

since $|f(x)| \leqq 1$.

Since the cosine is negative in the intervals $(-\pi, -\pi/2)$ and $(\pi/2, \pi)$ and positive in the remaining interval $(-\pi/2, \pi/2)$,

$$\int_{-\pi}^{\pi} |\cos x|\, dx = \int_{-\pi}^{-\pi/2} (-\cos x)dx + \int_{-\pi/2}^{\pi/2} \cos x\, dx$$

$$+ \int_{\pi/2}^{\pi} (-\cos x)dx = 4.$$

Therefore, $|a_1| \leqq 4/\pi$. However, if $f_1(x) = -1,\ -\pi < x < -\pi/2$; $f_1(x)$

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[2] American Journal of Mathematics, vol. 61 (1939).