A STIELTJES INTEGRAL EQUATION[1]

F. G. DRESSEL

Introduction. By making use of the notion of derivative taken with respect to a function of bounded variation, solutions of the Young-Stieltjes integral equations of the following type

$$F(x) = M(x) + \lambda \int_0^1 H(x, y)dF(y) \tag{1}$$

are obtained in this paper. Since the integration by parts formula is not valid for Young-Stieltjes integrals, equation (1) cannot be immediately changed to the equation

$$f(x) = m(x) + \lambda \int_0^1 f(y)dK(x, y). \tag{2}$$

Fischer[2] in his treatment of equation (2) imposed three conditions on $K(x, y)$ to obtain solutions. In the second part of this paper, we show that one of these conditions is sufficient to insure the existence of a bounded monotone increasing function $g(y)$ such that

$$| K(x, y_2) - K(x, y_1) | \leqq | g(y_2) - g(y_1) |. \tag{3}$$

It has been shown[3] that if condition (3) holds, then equation (2) can be changed into an ordinary Fredholm equation. Thus it appears that only one of the three conditions imposed on $K(x, y)$ by Fischer is needed to insure the solution of (2).

Before handling equation (1), let us note that if $g(y_1) < g(y_2)$ for $y_1 < y_2$, we may apply the transformation[4]

$$\beta(s) = \limsup E_y[s \geqq g(y)], \qquad g(0) \leqq s \leqq g(1), \tag{4}$$

and send the integral equation

$$v(x) = w(x) + \lambda \int_0^1 G(x, y)v(y)dg(y) \tag{5}$$

into the Fredholm equation

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[2] C. A. Fischer, *The Fredholm theory of the Stieltjes equation*, Annals of Mathematics, (2), vol. 25 (1923–1924), pp. 124–158.

All functions used in this paper are assumed to be measurable Borel.

[3] F. G. Dressel, *A note on Fredholm-Stieltjes integral equations*, this Bulletin, vol. 44 (1938), pp. 434–437.

[4] $E_y[s \geqq g(y)]$ designates the set of values of y such that $s \geqq g(y)$.