ON THE MEAN VALUES OF AN ANALYTIC FUNCTION[1]

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This note contains improvements on the results in two recent papers by Nehari.[2]

The first paper shows that if $f(z)$ is regular for $|z|<1$, and if the mean of $|f(z)|$ on the circle $|z|=r$ is less than or equal to 1 for each $r<1$, then the mean of $|f(z)|^2$ on $|z|=r$ is less than or equal to 1 for $r\leqq 6^{-1/2}$. We shall show that the conclusion is true for $r\leqq 2^{-1/2}$, but not always for a larger value of r. *More generally, we shall show that the mean of* $|f(z)|^p$ *on* $|z|=r$ *is less than or equal to* 1 *for* $r\leqq p^{-1/2}$ (*where* $p>1$ *is an integer*), *and that this result is the best possible.*

It will be sufficient to prove that if $g(z)$ is a function which is regular for $|z|\leqq 1$ and different from 0 for $|z|<1$, and such that the mean of $|g(z)|$ on $|z|=1$ is less than or equal to 1, then the mean of $|g(z)|^p$ on $|z|=r$ is less than or equal to 1 for $r\leqq p^{-1/2}$. For suppose $0<R<1$, and put

$$g(z) = f(Rz):\prod_{\nu=1}^{n} \frac{z-\alpha_\nu}{1-\bar{\alpha}_\nu z},$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the zeros of $f(Rz)$ in $|z|<1$. We note that $|g(z)|=|f(Rz)|$ for $|z|=1$, while $|g(z)|>|f(Rz)|$ for $|z|<1$. The function $g(z)$ evidently satisfies the above conditions. From the conclusion that the mean of $|g(z)|^p$ on $|z|=r$ is less than or equal to 1 for $r\leqq p^{-1/2}$, we see that the mean of $|f(Rz)|^p$ on $|z|=r$ is not greater than 1 for $r\leqq p^{-1/2}$, or that the mean of $|f(z)|^p$ on $|z|=r$ is not greater than 1 for $r\leqq Rp^{-1/2}$. The desired result follows by letting $R\to 1$.

We have to show that from the hypothesis $(1/2\pi)\int_0^{2\pi}|g(e^{i\theta})|\,d\theta\leqq 1$ the conclusion

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leqq 1, \qquad \text{for } r\leqq p^{-1/2},$$

follows. Now since $g(z)\neq 0$ for $|z|<1$, we may put $g(z)=h(z)^2$, where $h(z)$ is regular for $|z|<1$. If we put

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad h(z)^p = \sum_{n=0}^{\infty} c_n z^n,$$

[1] Presented to the Society December 2, 1939.

[2] Comptes Rendus de l'Académie des Sciences, Paris, vol. 206 (1938), pp. 1943–1945; vol. 208 (1939), pp. 1785–1787. My results were obtained during a summer (1939) spent at Stanford University. The two papers mentioned were called to my attention by Professor Szegö.