A THEOREM CONCERNING CLOSED AND COMPACT POINT SETS WHICH LIE IN CONNECTED DOMAINS[1]

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The purpose of this paper is to show that the following theorem holds in any space which satisfies Axioms 0, 1, and 2 of R. L. Moore's *Foundations of Point Set Theory*.[2]

If g denotes a point set, $\bar{g}$ will be used to denote the set g together with all its limit points. For each positive integer n, G_n will denote the collection G_n of Axiom 1.

THEOREM. *If M is a closed and compact subset of a connected domain D, then there exists a compact continuum containing M and lying in D.*

PROOF. For each point P of D, there exists a region g_P of G_1 containing P such that $\bar{g}_P$ is a subset of D. By Axiom 2, there exists a connected domain d_P containing P which is a subset of g_P. Let U_1 denote the collection of all domains d_P for each point P of D. The point set M is closed and compact, and hence, by Theorem 22 of Chapter I, it is covered by a finite subcollection W_1 of U_1. By Theorem 77 of Chapter I, for each pair of domains x and y of W_1 there exists a simple chain xy whose links are domains of U_1 and whose first and last links are x and y respectively. Let V_1 denote the collection of all domains v such that for some two domains x and y of W_1, v is a link of the chain xy. The sum of all the domains of the finite collection V_1 is a connected domain D_1. Similarly, there exists a finite collection V_2 of connected domains such that if v is any domain of V_2, then $\bar{v}$ is a subset of some region of G_2 and of some domain of V_1, and such that the sum of the domains of V_2 is a connected domain D_2. This process can be continued. Thus there exists an infinite sequence $V_1, V_2, V_3, \cdots$ such that, for each n, (1) V_{n+1} is a finite collection of connected domains such that if v is any one of them then $\bar{v}$ is a subset of some region of G_{n+1} and of some domain of V_n and of D, and (2) the sum of all the domains of V_n is a connected domain D_r containing M. By Theorems 79 and 80 of Chapter I, the set of all points common to all the sets of the sequence $D_1, D_2, D_3, \cdots$ is a compact continuum, and it contains M and lies in D.

A modification of this argument proves this theorem for a space which satisfies Axioms 0 and 1 and is locally arcwise connected.

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[1] Presented to the Society, February 24, 1940.

[2] American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. All references are to this book.