A NEW APPROACH TO THE CRITICAL VALUE THEORY

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The well known inequalities of M. Morse have as algebraical foundation the rank-equations

$$\text{(I)}\qquad rB_i(\Sigma - \Sigma_1) = rB_i(\Sigma) - rB_i(\Sigma_1) + rD_i(\Sigma_1, \Sigma) + rD_{i-1}(\Sigma_1, \Sigma), \qquad i = 0, 1, 2, \cdots .$$

These formulas hold for any topological group-system Σ, any subsystem Σ_1 of Σ, and the difference-system $\Sigma - \Sigma_1$. (W. Mayer, *Topologische Gruppensysteme*, Monatshefte für Mathematik und Physik, vol. 47 (1938); henceforth referred to as M, TG.) Here $B_i(\Sigma)$ denotes the i-dimensional Betti group of Σ, while $B_i(\Sigma_1)$ and $B_i(\Sigma - \Sigma_1)$ are these groups for Σ_1 and $\Sigma - \Sigma_1$ respectively. The symbol $r(\ )$, of course, stands for the rank of the group in the parentheses. By $D_i(\Sigma_1, \Sigma)$ we mean the subgroup of $B_i(\Sigma_1)$ containing all the classes of this group whose elements bound in Σ.

The formula (I) was first derived for the case of a complex in Lefschetz' *Topology*, 1930 (p. 150), and independently for the complex modulo 2 by J. Rybarz, Monatshefte für Mathematik und Physik (1931).

In the generality needed here the proof of (I) is given in M, TG (pp. 54–57), under the assumption, of course, that all the ranks appearing in (I) are finite, since otherwise the formula would be meaningless. But the proof there given shows also that

(a) $rB_i(\Sigma - \Sigma_1) = \infty$ implies that either $rB_i(\Sigma)$ or $rD_{i-1}(\Sigma_1, \Sigma)$, or both, are infinite;

(b) $rB_i(\Sigma - \Sigma_1)$ finite implies $rD_{i-1}(\Sigma_1, \Sigma)$ finite, and if in addition $rB_i(\Sigma_1)$ is finite then $rB_i(\Sigma)$ is finite too; and

(c) $rB_i(\Sigma - \Sigma_1) = 0$ implies $rD_{i-1}(\Sigma_1, \Sigma) = 0$ and if in addition $rB_i(\Sigma_1)$ is finite, then $rB_i(\Sigma_1) = rB_i(\Sigma) + rD_i(\Sigma_1, \Sigma)$.

As an immediate consequence of equations (I) we notice the inequality

$$\text{(I')}\qquad rB_i(\Sigma) \leqq rB_i(\Sigma_1) + rB_i(\Sigma - \Sigma_1),$$

which is true whenever the terms on the right are finite (remark (b)) and trivial otherwise. The next step in attaining the Morse inequalities is the application of (I) to $m+2$ topological group-systems satisfying the inclusion relations

$$(1)\qquad \Sigma_m \supset \Sigma_{m-1} \supset \cdots \supset \Sigma_0 \supset \Sigma_{-1}$$