TOTAL REGULARITY OF GENERAL TRANSFORMATIONS[1]

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A method of summation is said to be regular if it assigns to every convergent series its actual value. If it also assigns the value $+\infty$ (or $-\infty$) to every series which diverges to $+\infty$ (or $-\infty$) it is said to be totally regular. The conditions for regularity are well known, and those for total regularity have been worked out for triangular matrix transformations by W. A. Hurwitz.[2] We here obtain necessary and sufficient conditions for total regularity for a more general type of transformation. (The former conditions, though still sufficient, are not necessary.)

Suppose $x_1, x_2, x_3, \cdots$ is the sequence of partial sums of the original series which is assumed real. A value Y is assigned to this sequence in the following way:

$$Y = \lim_{D(t)\to 0} y(t); \qquad y(t) = \sum_{k=1}^{\infty} a_k(t) x_k.$$

t is a variable ranging over some point set, $D(t)$ is a positive real function, and the functions $a_k(t)$ are real, but not necessarily continuous. We assume the transformation is regular so that the three Silverman-Toeplitz conditions are satisfied:

(1) $\sum_{k=1}^{\infty} |a_k(t)|$ is bounded for $D(t)$ sufficiently small;

(2) $\lim_{D(t)\to 0} \sum_{k=1}^{\infty} a_k(t) = 1$;

(3) $\lim_{D(t)\to 0} a_k(t) = 0$ for all k.

We then ask when Y will be positively infinite if $\lim_{k\to\infty} x_k = +\infty$. (We demand that for $D(t)$ sufficiently small $y(t)$ will be defined although it may be positively infinite.)

First it may be seen that for sufficiently advanced t (that is, t for which $D(t)$ is sufficiently small) there can be only a finite number of negative coefficients $a_k(t)$ in each row (that is, for each t) if the transformation is to be totally regular. Otherwise a sequence t_n with $D(t_n)\to 0$ could be picked out such that for each t_n there would be an infinite number of negative coefficients. Then a sequence x_k could be defined so that $x_k \to \infty$ and

$$\sum_{k=1}^{\infty} a_k(t_n) x_k$$

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[2] W. A. Hurwitz, Proceedings of the London Mathematical Society, vol. 26 (1927), p. 231.