THE SPACES L^p WITH $0<p<1$[1]

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If we have a Lebesgue measurable set E in any n-dimensional euclidean space, and have any positive number q, we define $L^q(E)$ to be the class of all real-valued Lebesgue measurable functions f on E for which $\int_E |f|^q < \infty$. As is well known, whenever $q \geqq 1$ the class of functions $L^q(E)$ is a Banach space with the norm $\|f\|_q = (\int_E |f|^q)^{1/q}$. When $0<p<1$, the function $\|f\|_p$ no longer satisfies the triangle inequality $\|f_1+f_2\|_p \leqq \|f_1\|_p + \|f_2\|_p$ but in general only the weaker condition $\|f_1+f_2\|_p \leqq 2^\nu[\|f_1\|_p + \|f_2\|_p]$, where $\nu=(1-p)/p$. (This can be shown by considering the function $(1+x^p)/(1+x)^p$.) If we consider such an L^p space as a linear topological space in which the neighborhoods of a point f_0 are the spheres of radius $\epsilon>0$, $E_{f \varepsilon L^p}[\|f-f_0\|_p < \epsilon]$, it follows from theorems of Hyers and Wehausen[3] that this topology can be given by an equivalent Fréchet metric. This suggests that while many theorems on Banach spaces which can be applied to the spaces $L^p(E)$ with $p \geqq 1$ may fail to hold in those spaces with $0<p<1$, there may still remain many theorems on Fréchet spaces and pseudo-normed spaces which may be applicable. However, Theorem 1 shows that almost no results depending on the use of linear (that is, additive and continuous) functionals can be usefully applied in these spaces.

THEOREM 1. *Any linear functional on $L^p(E)$ is identically zero.*

The proof of this and of some additional results is given in a series of lemmas using a more general set of assumptions. We assume as a background some knowledge of the first chapter of Saks[4] book, in which he deals with what he calls "the integral," a completely additive integral having many of the properties of the Lebesgue integral. We consider a set Y of elements y, an additive family[5] $\mathfrak{X}$ of subsets of Y and an additive, non-negative[6] set-function μ on $\mathfrak{X}$ such that $\mu(Y)<\infty$. This last condition will be imposed from here until we

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[3] D. H. Hyers, *A note on linear topological spaces*, this Bulletin, vol. 44 (1938), pp. 76–80, and J. V. Wehausen, *Transformations in linear topological spaces*, Duke Mathematical Journal, vol. 4 (1938), pp. 157–169.

[4] S. Saks, *Theory of the Integral*, Warsaw, 1937.

[5] A class $\mathfrak{X}$ of subsets of Y is called an additive family if (a) $X \varepsilon \mathfrak{X}$ implies $Y-X \varepsilon \mathfrak{X}$, and (b) $X_n \varepsilon \mathfrak{X}$ implies $\sum_{n<\infty} X_n \varepsilon \mathfrak{X}$.

[6] A set-function μ is additive if whenever X_n are disjoint sets of $\mathfrak{X}$ then $\mu(\sum_{n<\infty} X_n) = \sum_{n<\infty} \mu(X_n)$; μ is non-negative if $\mu(X) \geqq 0$ for every $X \varepsilon \mathfrak{X}$.