CLOSURE OF PRODUCTS OF FUNCTIONS[1]

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This note presents some natural theorems on the characterizations of certain closed (*or complete*) sets of functions with separable variables. In order to motivate the developments of the paper we treat a simple case first in elaborate detail. The proof is so formulated that it holds with trifling modifications for the more general situations in Theorems 3 and 4. The result in Theorem 5 belongs to a slightly different range of ideas.

Let $s\sim(s_1, \cdots, s_m)$ and $t\sim(t_1, \cdots, t_n)$ here stand for points in the euclidean spaces R_m and R_n. The term "interval" designates the generalized rectangular parallelopipedon open on the left.[2] We shall make use of the intervals $I_s \subset R_m$, $I_t \subset R_n$ and $I_2 = I_s \times I_t \subset R_{n+m}$. We are first interested in $L_2(I)$, the space of complex valued functions of summable square over I. The norm and scalar product are defined as usual by

$$(1) \qquad \|f(s, t) - g(s, t)\| = \left[\int_{I_t}\int_{I_s} |f(s, t) - g(s, t)|^2 dI_s dI_t\right]^{1/2},$$

$$(2) \qquad (f(s, t), g(s, t)) = \int_{I_t}\int_{I_s} f(s, t)\bar{g}(s, t) dI_s dI_t,$$

where $\bar{g}(s, t)$ is the conjugate of $g(s, t)$. The subscript I_s or I_t will indicate that the left-hand functionals are on the corresponding intervals.

We shall understand closure of the sequence of functions[3] $\{\phi_\gamma(t)\psi_\mu(s)\}$, $\gamma; \mu = 0, 1, \cdots$, to mean that for every $f(s, t) \in L_2(I_2)$ and arbitrary $\epsilon > 0$ there exists a finite sequence of complex constants $\{\beta_{\gamma\mu}\}$ and integers A and B such that

$$(3) \qquad \left\|f(s, t) - \sum_0^A \sum_0^B \beta_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s)\right\| < \epsilon.$$

It is well known that with the adjunction of the scalar product defined in (2), $L_2(I_2)$ is a complex Hilbert space and that closure and completeness are equivalent concepts.

THEOREM 1. *If* $\{\phi_\gamma(t)\psi_\mu(s)\}$, $\gamma, \mu = 0, 1, \cdots$, *is a sequence of com-*

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[2] S. Saks, *Theory of the Integral*, English edition, p. 57.

[3] Curly brackets, { }, will always denote sequences.