ABELIAN GROUPS THAT ARE DIRECT SUMMANDS OF EVERY CONTAINING ABELIAN GROUP[1]

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It is a well known theorem that an abelian group G satisfying $G=nG$ for every positive integer n is a direct summand of every abelian group H which contains G as a subgroup. It is the object of this note to generalize this theorem to abelian groups admitting a ring of operators, and to show that the corresponding conditions are not only sufficient but are at the same time necessary. Finally we show that every abelian group admitting a ring of operators may be imbedded in a group with the above mentioned properties; and it is possible to choose this "completion" of the given group in such a way that it is isomorphic to a subgroup of every other completion.

Our investigation is concerned with abelian groups admitting a ring of operators. A ring R is an abelian group with regard to addition, its multiplication is associative, and the two operations are connected by the distributive laws. As the multiplication in R need not be commutative, we ought to distinguish left-, right- and two-sided ideals. Since, however, only left-ideals will occur in the future, we may use the term "ideals" without fear of confusion. Thus an ideal in R is a non-vacuous set M of elements in R with the property:

If m', m'' are elements in M, and if r', r'' are elements in R, then $r'm' \pm r''m''$ is an element in M.

An abelian group G whose composition is written as addition is said to admit the elements in the ring R as operators (or shorter: G is an *abelian group over* R), if with every element r in R and g in G is connected their uniquely determined product rg so that this product is an element in G and so that this multiplication satisfies the associative and distributive laws. If G is an abelian group over R, then its subgroups M are characterized by the same property as the ideals M in R.

We assume finally the existence of an element 1 in R so that $g=1g$ for every g in G and $r\cdot 1=1\cdot r=r$ for every r in R.

If x is any element in the abelian group G over R, then its *order* $N(x)$ consists of all the elements r in R which satisfy $rx=0$. One verifies that every order $N(x)$ is an ideal in R, and that $N(x)=R$ if, and only if, $x=0$.

If M is an ideal in R, and if x is an element in G, then a subgroup

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