ON REARRANGEMENTS OF SERIES[1]

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1. **Introduction.** Let E denote the metric space in which a point x is a permutation $x_1, x_2, x_3, \cdots$ of the positive integers and the distance (x, y) between two points $x \equiv \{x_1, x_2, \cdots\}$ and $y \equiv \{y_1, y_2, \cdots\}$ of E is given by the Fréchet formula

$$(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

The space E is of the second category (Theorem 2).

Let $c_1 + c_2 + \cdots$ be a convergent series of real terms for which $\sum |c_n| = \infty$. To simplify typography, we write $c(n)$ for c_n. To each $x \varepsilon E$ corresponds a rearrangement $c(x_1) + c(x_2) + \cdots$ of the series $\sum c_n$. By a well known theorem of Riemann, $x \varepsilon E$ exists such that $c(x_1) + c(x_2) + \cdots$ converges to a preassigned number, diverges to $+\infty$ or to $-\infty$, or oscillates in a preassigned fashion.

The set A of $x \varepsilon E$ for which $c(x_1) + c(x_2) + \cdots$ converges is therefore a proper subset of E, and M. Kac has proposed the problem of determining whether $E - A$ is of the second category. The following theorem shows not only that A is of the first category (and hence that $E - A$ is of the second category) but also that the set of $x \varepsilon E$ for which the series $c(x_1) + c(x_2) + \cdots$ has unilaterally bounded partial sums is of the first category.

THEOREM 1. *For each $x \varepsilon E$ except those belonging to a set of the first category,*

$$\liminf_{N \to \infty} \sum_{n=1}^{N} c(x_n) = -\infty, \qquad \limsup_{N \to \infty} \sum_{n=1}^{N} c(x_n) = \infty.$$

2. **Proof of Theorem** 1. The fact that the "coordinates" x_n and y_n of two points x and y of E are integers implies roughly that, if N is large, then $x_n = y_n$ for $n = 1, 2, \cdots, N$ if and only if (x, y) is near 0. To make this precise, let $x \varepsilon E$, $r > 0$, and let $S(x, r)$ denote the set of points y such that $(x, y) < r$, so that $S(x, r)$ is an open sphere with center at x and radius r. It is easy to show that if x and y are two points of E such that $y \varepsilon S(x, 2^{-N-1})$ then $x_n = y_n$ when $n = 1, 2, \cdots, N$; and that if x and y are such that $x_n = y_n$ when $n = 1, 2, \cdots, N$ then $y \varepsilon S(x, 2^{-N})$.

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