NEUTRAL ELEMENTS IN GENERAL LATTICES[1]

GARRETT BIRKHOFF

1. **Introduction.** O. Ore has defined "neutral" elements in modular lattices as elements a satisfying $a \cap (x \cup y) = (a \cap x) \cup (a \cap y)$ for all x, y and dually.[2] In the case of complemented modular lattices, the neutral elements compose the "center" in J. von Neumann's theories of continuous geometries and regular rings—that is, the set of elements having unique complements.[3]

The purpose of the present note is to extend the notion of neutral elements to general lattices. More precisely, call an element a of a lattice "neutral" if and only if every triple $\{a, x, y\}$ generates a distributive sublattice. It is proved that the neutral elements of any lattice L form a distributive sublattice, consisting of the elements carried into $[I, O]$ under isomorphisms of L with sublattices of direct products. Actually, this sublattice is the intersection of the maximal distributive sublattices of L.

Further, complements of neutral elements, when they exist, are unique and neutral. The sublattice of complemented neutral elements may be called the "center" of a lattice: it consists of those elements carried into $[I, O]$ under isomorphisms of L with direct products.

2. **Fundamental definition.** We define an element a of a lattice L to be "neutral" if and only if every triple $\{a, x, y\}$ generates a distributive sublattice of L.

LEMMA 1. *If a is "neutral," then the dual correspondences $x \rightarrow x \cap a$ and $x \rightarrow x \cup a$ are endomorphisms[4] of L.*

PROOF. By definition, $(x \cup y) \cap a = (x \cup a) \cap (y \cup a)$ and $(x \cap y) \cap a = (x \cap a) \cap (y \cap a)$, and dually. We note that this condition, which is sufficient to guarantee neutrality in the case of modular lattices, does not guarantee neutrality in general—see the graph below.

---

[1] Presented to the Society, September 8, 1939.

[2] O. Ore, *On the foundations of abstract algebra* I, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437. For the definitions of lattices and modular lattices (called by Ore structures and Dedekind structures), as well as of sublattice, distributive lattice, O, I, and so on, compare the author's *Lattices and their applications*, this Bulletin, vol. 44 (1938), pp. 793–800—or the author's *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, 1940.

[3] J. von Neumann, *Lectures on Continuous Geometries*, Princeton, 1935–1936. Cf. also R. P. Dilworth, *Note on complemented modular lattices*, this Bulletin, vol. 45 (1939), pp. 74–76.

[4] We define an endomorphism as a homomorphism of L with itself.