A REMARK CONCERNING SUFFICIENCY THEOREMS FOR THE PROBLEM OF BOLZA[1]

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In recent years there have appeared several sufficiency theorems for minima in Bolza problems in which there is no assumption that the curve in question is normal. However, it is assumed in all these theorems that the multiplier rule, the transversality conditions and the strengthened conditions of Weierstrass, Clebsch and Jacobi hold for a set of multipliers with $\lambda_0 > 0$; whereas, among the necessary conditions the multiplier rule and the Weierstrass and Clebsch conditions have been shown[2] to hold for multipliers with $\lambda_0 \geqq 0$.

In the present note we close this slight gap by showing that the sufficiency theorems hold if we assume λ_0 non-negative instead of positive. The only case needing discussion is $\lambda_0 = 0$. In this case we show that the curve satisfying the hypothesis is isolated; no neighboring curve satisfies the side-equations and end-conditions. The curve is thus a minimizing curve in a trivial sense. Less trivially, we show[3] that the curve is a normal proper minimizing curve for a Mayer problem in which the end-conditions consist of a subset of the original end-conditions and the function to be minimized is a linear combination of the original end-functions.

The problem is that of minimizing a functional

$$J[y] = g(x_1, y(x_1), x_2, y(x_2)) + \int_{x_1}^{x_2} f(x, y(x), y'(x))dx \tag{1}$$

in the class of curves $y^i = y^i(x)$, $x_1 \leqq x \leqq x_2$, $i = 1, \cdots, n$, which satisfy certain differential equations

$$\phi_\alpha(x, y(x), y'(x)) = 0, \qquad \alpha = 1, \cdots, m < n, \tag{2}$$

and certain end-conditions

$$\psi_\mu(x_1, y(x_1), x_2, y(x_2)) = 0, \qquad \mu = 1, \cdots, p \leqq 2n + 2. \tag{3}$$

We suppose the usual conditions[4] of continuity and differentiability

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[2] E. J. McShane, *On multipliers for Lagrange problems*, American Journal of Mathematics, vol. 61 (1939), pp. 809–819.

[3] At the suggestion of Professor M. R. Hestenes.

[4] See, e.g., G. A. Bliss, *The problem of Bolza in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 261–273.