SYSTEMS OF ORTHOGONAL POLYNOMIALS ON CERTAIN ALGEBRAIC CURVES[1]

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1. **Introduction.** A system of orthogonal polynomials in two real variables x and y can be defined with respect to any domain of integration in the xy plane and any nonnegative weight function which has a positive integral over that domain. Although a general study of the formal properties of such systems has been made,[2] the corresponding problem of convergence has been adequately dealt with only for cases which reduce almost immediately to familiar problems in a single variable. The purpose of this paper is to give more substantial examples of systems for which it is possible, by special methods, to present an account of convergence which is fully comparable, at least in some of its main features, with such highly developed theories as those of Fourier and Legendre series.

The proofs, as in the case of series of orthogonal polynomials in one variable with a more or less general weight function, can be made to depend on properties of boundedness. The method used here for obtaining these properties is to establish relationships between the systems in two variables considered and systems in one variable whose properties are well known.

The domain of integration to which primary consideration is to be given is the perimeter of the square whose sides are segments of the lines $x=\pm 1$, $y=\pm 1$. Two other domains will be dealt with briefly in a concluding section.

The square contour will be denoted by the letter C. We shall take the following sequence as the basis for the construction of a set of orthogonal polynomials on C: 1, $x+y$, $x-y$, xy, x^2+y^2, x^2-y^2, x^2y+xy^2, x^2y-xy^2, x^3+y^3, x^3-y^3, $\cdots$, $x^{n-1}y+xy^{n-1}$, $x^{n-1}y-xy^{n-1}$, x^n+y^n, x^n-y^n, $\cdots$. The terms of this sequence have the property that any finite number of them are linearly independent on C; and also, by means of the identity $x^2y^2-x^2-y^2+1\equiv 0$ which holds everywhere on C, any polynomial in x and y can be expressed on C as a linear combination of them. If $\rho(x, y)$ is a function which is positive almost everywhere on C, we can multiply each member of the sequence by $\rho^{1/2}$ and apply the Schmidt process of orthogonalization,

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[2] D. Jackson, (1) *Formal properties of orthogonal polynomials in two variables*, Duke Mathematical Journal, vol. 2 (1936), pp. 423–434. (2) *Orthogonal polynomials on a plane curve*, Duke Mathematical Journal, vol. 3 (1937), pp. 228–236.