

**NOTE ON THE COMPUTATION OF THE DIFFERENCES OF
THE $\text{Si}(x)$, $\text{Ci}(x)$, $\text{Ei}(x)$ AND $-\text{Ei}(-x)$ FUNCTIONS**

MILTON ABRAMOWITZ

In a recent article,¹ Lowan reached the conclusion that the second differences of the functions $\text{Si}(x)$, $\text{Ci}(x)$ and $\text{Ei}(x)$ for values of x from 0.1 to 2 at intervals of 0.0001 may be approximated to 12 decimal places by certain expressions in closed form, which may be computed with the aid of tabulated functions.

The above functions and also $-\text{Ei}(-x)$ were computed to 12 decimal places by adding an appropriate number of terms in their power series expansion.² In order to check the accuracy of the computations, the second differences of the computed values were required to agree to 12 decimal places with the values of the approximate expressions mentioned above.

The functions $\text{Ci}(x)$, $\text{Ei}(x)$, $-\text{Ei}(-x)$ have a logarithmic singularity at the origin. For this reason Lowan's conclusion could not be extended to the interval $0 < x < 0.1$.

If $\phi(x)$ represents any of the functions $\text{Ci}(x) - \log_e x$, $\text{Ei}(x) - \log_e x$, $-\text{Ei}(-x) + \log_e x$, it will be shown that the second difference of $\phi(x)$ may be approximated by $h^2\phi''(x)$ to 12 decimal places. Let

$$R(x) = [\phi(x+h) - 2\phi(x) + \phi(x-h)] - h^2\phi''(x).$$

Substituting for $\phi(x+h)$ and $\phi(x-h)$ their Taylor expansions, we get

$$R(x) = 2 \left[\frac{h^4}{4!} \phi^4(x) + \frac{h^6}{6!} \phi^6(x) + \dots \right]$$

whence

$$(A) \quad |R(x)| < 2 \left[\frac{h^4}{4!} \{\phi^4(x)\} + \frac{h^6}{6!} \{\phi^6(x)\} + \dots \right]$$

where $\{\phi^{2k}(x)\}$ is an upper bound of the modulus of the $2k$ th derivative.

¹ A. N. Lowan, *On the computation of second differences of the $\text{Si}(x)$, $\text{Ei}(x)$, and $\text{Ci}(x)$ functions*, this Bulletin, vol. 45 (1939), pp. 583-588. This will be referred to as A.N.L.

² This work is done by the New York City Work Projects Administration project for the Computation of Mathematical Tables, O. P. #765-97-3-10, under the sponsorship of Dr. Lyman J. Briggs, Director of the National Bureau of Standards. The author wishes to express his appreciation to the W.P.A. and to the sponsor of this project for permission to publish these results.