ON PERFECT SUMMABILITY OF DOUBLE SEQUENCES

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1. Introduction. The purpose of the present note is to point out that the results of Banach on perfect summability of simple sequences[1] may be extended to certain cases of double sequence summability. The main result obtained is embodied in the theorem of §3.

It will be convenient to begin with the following definitions and notations. We denote by c the class of all real double sequences $x \equiv \{\xi_{kl}\}$ for which the *principal limit* $\lim_{k,l} \xi_{kl} \equiv \xi$ exists finite; by rc, the subclass of c for which the *row limits* $\lim_l \xi_{kl} \equiv \xi_k^r$, $(k=0, 1, 2, \cdots)$ and the *column limits* $\lim_k \xi_{kl} \equiv \xi_l^c$, $(l=0, 1, 2, \cdots)$, exist finite; and by $rcrn$, the subclass of rc throughout which $\xi_k^r = \xi_l^c = \xi = 0$, $(k, l=0, 1, 2, \cdots)$. With the conventional definitions of addition and multiplication by a constant the classes rc and $rcrn$ are linear spaces, and they become Banach spaces upon introduction of the norm $\|x\| \equiv \sup_{k,l} |\xi_{kl}|$.

Let $A \equiv (a_{ijkl})$ be a given infinite matrix of real numbers. We shall be concerned with transformations of the form

$$A_{ij}(x) \equiv \sum_{k,l} a_{ijkl}\xi_{kl}, \qquad i, j = 0, 1, 2, \cdots, \tag{1.1}$$

on the elements $x \equiv \{\xi_{kl}\}$ of rc. More precisely, if $A_{ij}(x)$, $(i, j=0, 1, 2, \cdots)$, exists for every $x \in rc$ and if the corresponding sequence $\{A_{ij}(x)\}$ belongs to rc $[c]$, one says that rc is transformed into rc $[c]$ by the method of summability corresponding to the matrix A, or simply, by the method A. If $A(x) \equiv \lim_{i,j} A_{ij}(x) = \xi$ for every $x \in rc$, the method is called *regular*. A regular transformation of rc into itself is called *completely regular* if it is also *regular by rows and columns*, that is to say, if $A_i^r(x) \equiv \lim_j A_{ij}(x) = \xi_i^r$, $(i=0, 1, 2, \cdots)$, and $A_j^c(x)$ $\lim_i A_{ij}(x) = \xi_j^c$, $(j=0, 1, 2, \cdots)$, for every $x \in rc$.

For later reference we recall that the condition

$$\sup_{i,j} \sum_{k,l} |a_{ijkl}| \equiv M < \infty \tag{1.2}$$

is necessary[2] in order that A transform $rcrn$ into rc.

If the system of equations

$$\sum_{k,l} a_{ijkl}\xi_{kl} = \eta_{ij}, \qquad i, j = 0, 1, 2, \cdots, \tag{1.3}$$

[1] Banach, *Théorie des Opérations Linéaires*, pp. 90–95.

[2] See Hamilton, *Transformations of multiple sequences*, Duke Mathematical Journal, vol. 2 (1936), pp. 29–60; in particular, p. 42, #5.