CONCERNING SIMILARITY TRANSFORMATIONS OF LINEARLY ORDERED SETS[1]

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1. **Introduction.** As is well known, two linearly ordered sets A and B are said to be similar if there exists a 1-1 correspondence between their elements which preserves order. A function f which defines such a 1-1 correspondence may be called a similarity transformation on A to B. In this note we consider two problems concerning similarity transformations which do not seem to have received attention heretofore. The first problem is the following:

(A) *Is it true that every infinite ordered set is similar to a proper subset of itself?*[2]

Before stating the second problem we recall a classical theorem concerning well-ordered sets.[3]

THEOREM. *If the set A is well-ordered, and if f is any similarity transformation on A to a subset of A, then $f(a) \geqq a$ for every a in A.*

It is natural to inquire whether this theorem characterizes well-ordered sets—and this is our second problem; more explicitly:

(B) *Let A be a linearly ordered set such that if f is any similarity transformation on A to a subset of A then $f(a) \geqq a$ for every a in A. Is it true that any such set A is well-ordered?*

We will demonstrate that if the set A is denumerable, then the answer to both questions is in the affirmative. An example will then be constructed to show that these conclusions need not hold if the set A is nondenumerable.

2. **The denumerable case.** We obtain first the following result:

THEOREM 1. *Every denumerably infinite linearly ordered set A contains a proper subset A' to which it is similar.*

PROOF. For any two elements a and b of A, we will say that a and b are *congruent* if either $a=b$ or if there is only a finite number of ele-

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[2] This question is a natural one, in view of the familiar definition of an infinite set as one which is *equivalent* to a proper subset of itself.

[3] For theorems mentioned in this paper one may refer to Hausdorff's *Grundzüge der Mengenlehre*, 1st edition, 1914, or to Sierpiński's *Leçons sur les Nombres Transfinis*, 1928.