# INDEFINITELY DIFFERENTIABLE FUNCTIONS WITH PRESCRIBED LEAST UPPER BOUNDS[1]

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1. **Introduction.** Let $F(x)$ be a real indefinitely differentiable function of the real variable $x$ defined on the interval $a \leqq x \leqq b$, and let $M_n$ denote the least upper bound of $|F^{(n)}(x)|$ on that interval.[2] In this paper we shall establish sufficient conditions that there exist an indefinitely differentiable function $F(x)$ taking on certain prescribed $M_n$.

It is easy to see that $M_0$ and $M_1$ can be assigned arbitrarily. However, the first three upper bounds $M_0$, $M_1$, and $M_2$ must satisfy certain inequalities.[3] Let us consider the interval $(0, 1)$. Let $t_1$ be the value of $x$ for which $|F^{(1)}(x)|$ attains its maximum. Then

$$F(1) - F(t_1) = (1 - t_1)F^{(1)}(t_1) + (1/2!)F^{(2)}(\theta_1)(1 - t_1)^2,$$

where $t_1 < \theta_1 < 1$. And similarly

$$F(0) - F(t_1) = - t_1F^{(1)}(t_1) + (1/2!)F^{(2)}(\theta_2)t_1^2,$$

where $0 < \theta_2 < t_1$. On subtracting these equations we obtain

$$F^{(1)}(t_1) = F(1) - F(0) + (1/2!)\left\{F^{(2)}(\theta_2)t_1^2 - F^{(2)}(\theta_1)(1 - t_1)^2\right\},$$
$$M_1 \leqq 2M_0 + M_2/2!.$$

By the same procedure we can obtain for the interval $(0, a)$

$$M_1 \leqq 2M_0/a + M_2a/2!. \tag{1}$$

In the case of the interval $(0, \infty)$ we can replace (1) by a more precise inequality. Since $a$ is arbitrary, we can replace $a$ by the positive value which minimizes the right side of (1), and obtain

$$M_1 \leqq 2(M_0M_2)^{1/2}.$$

Ore[4] in a recent paper employed the results of W. Markoff to obtain certain inequalities connecting the least upper bounds of $|F^{(i)}(x)|$, $(1 \leqq i \leqq n)$, with those of $|F(x)|$ and $|F^{(n+1)}(x)|$ where $F(x)$ is a function with bounded $(n+1)$th derivative. For the first derivative the inequality (1) is slightly better than that obtained by Ore.

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[2] By $F^{(0)}(x)$ we shall mean $F(x)$.

[3] Hadamard, Comptes Rendus des Séances de la Société Mathématique de France, 1914, pp. 68–72; T. Carlman, *Les fonctions quasi analytiques*, Paris, 1926.

[4] O. Ore, *On functions with bounded derivatives*, Transactions of this Society, vol. 43 (1938), pp. 321–326.