A METHOD FOR PROVING CERTAIN ABSTRACT GROUPS TO BE INFINITE[1]

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1. **Introduction.** I have stated elsewhere[2] that the group (3, 3, 4; 4), defined by

$$R^3 = S^3 = (RS)^4 = (R^{-1}S^{-1}RS)^4 = 1,$$

is infinite. This fact will now be proved by showing that there is a factor group of order $24n^4$ for every positive integer n.

We shall find a closely related group of order $48n^4$, satisfying the relations $S^3 = T^2 = (ST)^8 = (S^{-1}TST)^6 = 1$, which have been studied by Brahana;[3] but there is no overlapping, since his "subgroup H" is not invariant in our case, although there still is an abelian invariant subgroup of index 48. In fact, it was the search for such a subgroup that led to the simple treatment given here.

Section 7 is inserted for its intrinsic interest, and can be omitted without impairing the proof of the main result (§8).

2. **A group of order n^4.** Consider the direct product of two cyclic groups of order n. Since the defining relations $M_1^n = M_2^n = M_1^{-1}M_2^{-1}M_1M_2 = 1$ imply $(M_1M_2)^n = 1$, they may be put into the form[4]

$$M_1^n = M_2^n = M_3^n = M_1M_2M_3 = M_3M_2M_1 = 1. \tag{1}$$

Hence the direct product of four cyclic groups of order n is defined by

$$\begin{aligned} M_i^n = M_1M_2M_3 = M_3M_2M_1 = N_j^n = N_1N_2N_3 = N_3N_2N_1 = 1, \\ M_iN_j = N_jM_i, \qquad i, j = 1, 2, 3. \end{aligned} \tag{2}$$

3. **A group of order $4n^4$.** These relations continue to hold when M_i is replaced by N_i, and N_j by M_j^{-1}. We now enlarge the group of order n^4 by adjoining an operator A, of period four, which transforms it according to this automorphism. The extra relations that have to be added to (2) are

$$A^4 = 1, \qquad A^{-1}M_iA = N_i, \qquad A^{-1}N_jA = M_j^{-1}.$$

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[1] Presented to the Society, September 6, 1938. The enumerative method described in the abstract (this Bulletin, 44-9-331) seems to be effective only in those cases where more orthodox methods are equally effective.

[2] Coxeter [**2**, p. 101, second footnote].

[3] Brahana [**1**].

[4] In the notation of Coxeter [**2**, p. 87], this is $(n, n, n; 1)$.