THE INEQUALITIES OF MORSE WHEN THE MAXIMUM TYPE IS AT MOST THREE[1]

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1. **Introduction.** The theory of critical points has been developed by Morse[2] [1, 2] by the use of combinatorial topology. In particular the theory developed by Morse is applicable to simple integral problems in the calculus of variations. The author, in his doctoral dissertation [3], studied the theory of types of extremals for a simple integral problem in the plane without the use of combinatorial topology. In the present paper a new and interesting property of extremal arcs joining two fixed points in the plane will be proved. This property makes possible a proof of the inequalities of Morse [3, p. 30] in the special case where the maximum type of each extremal arc is at most three. This is the first proof of the inequalities without the use of topology and without assuming the problem to be reversible [3, p. 25].

The hypotheses of this paper are those made in §1 of the dissertation [3] referred to above and likewise the notation used here is that of the earlier paper.

2. **Properties of the extremal arcs joining two fixed points of R.** In the paper mentioned above [3, p. 17] it was proved that every point 2 of R, which is not on an envelope arc of the family of extremal arcs though the point 1, is joined to 1 by $2r+1$ (r a positive integer) extremal arcs of which r are of odd type and $r+1$ are of even type, one of which at least is of type zero.

Consider now a point 2 of R which is not on an envelope arc of the family of extremals through the point 1. Let E_{a_0} designate an extremal arc joining the point 2 to 1 and such that the arc 12 of E_{a_0} is of type zero. It can be shown [3, p. 10] that the extremal arc E_a for $a>a_0$ and near a_0 and also for $a<a_0+2\pi$ and near $a_0+2\pi$ has no intersections with the arc 12 of E_{a_0}. Then as a increases from a_0 to $a_0+2\pi$ all intersections of the extremal arc E_a with the arc 12 of E_{a_0} which move onto this arc must move off again. In fact it can easily be proved [3, p. 18] that an intersection 3 of E_a with the arc 12 of E_{a_0} which moves onto the arc 12 of E_{a_0} for an extremal arc of odd (even) type must move off of the arc 12 of E_{a_0} for an extremal arc of even

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[2] The numbers in brackets here and elsewhere refer to the bibliography at the end of this paper.