A NOTE ON MEASURE FUNCTIONS IN A LATTICE[1]

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We give first an equivalent statement of the measurability criterion of Carathéodory[2] which is applicable to an arbitrary lattice. We then study the closure with respect to finite and denumerable sums and products of the subset of measurable elements of a *modular* lattice. The case of regular[3] "outer measure functions" is then briefly discussed. The elements of the theory of lattices are presupposed.[4]

Let us consider a lattice L on which is defined a real-valued function $\mu(a)$. The elements $a \in L$ which satisfy

$$\mu(a+b)+\mu(ab)=\mu(a)+\mu(b) \tag{1}$$

for every $b \in L$ will be called μ-measurable. The totality of μ-measurable elements will be denoted by $L(\mu)$.

REMARK 1. *If L is a Boolean algebra and $\mu(0)=0$, then $a \in L(\mu)$ if and only if $a \in L$ and satisfies the condition of Carathéodory,[5] that is,*

$$\mu(b)=\mu(ab)+\mu(b-ab) \tag{2}$$

for every $b \in L$. For, if $a \in L$ satisfies (1), *the equation* (1) *and*

$$\mu(a+(b-ab))+\mu(0)=\mu(a)+\mu(b-ab)$$

yield (2). *The converse is proved by Carathéodory.*[6]

THEOREM 1. *If L is a modular lattice, then $L(\mu)$ is a sublattice of L.*

PROOF. Let $a, c \in L(\mu)$, $b \in L$. We obtain successively

$$\begin{aligned}\mu(a+(c+b))+\mu(a(c+b)) &= \mu(a)+\mu(c+b)\\ &= \mu(a)+\mu(c)+\mu(b)-\mu(cb)\\ &= \mu(a+c)+\mu(b)+\mu(ac)-\mu(cb).\end{aligned}$$

Since $c \in L(\mu)$ we have

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[2] *Vorlesungen über Reelle Funktionen*, 2d edition, p. 246.

[3] Ibid., p. 258.

[4] See, for example, G. Birkhoff, *On the combination of subalgebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464; O. Ore, *On the foundations of abstract algebra* I, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437. The terminology and notation are those used by L. R. Wilcox and the author, *Metric lattices*, Annals of Mathematics, (2), vol. 40 (1939), pp. 309–327.

[5] Op. cit., p. 246.

[6] Ibid., p. 252.