SHORTER NOTICES

Die Gegenwärtige Lage in der mathematischen Grundlagenforschung. Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie. By Gerhard Gentzen. (Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, new series, no. 4.) Leipzig, Hirzel, 1938. 44 pp.

This is really two books printed in one volume. The first one, "Die gegenwärtige Lage in der mathematischen Grundlagenforschung," has also appeared in Deutsche Mathematik, vol. 3 (1938), pp. 255–268. The second one, "Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie," is a revision of Gentzen's famous paper in the Mathematische Annalen, vol. 112 (1936), pp. 493–565.

The first book is a well written summary of the present status of foundations, and contains one of the most lucid accounts of the Brouwer viewpoint that the present reviewer has seen. The distinction between the Brouwer and Hilbert schools is presented from the point of view of their treatment of the infinite. For Brouwer, who always insists on finite constructibility, the infinite exists only in the sense that he can at any time take a larger (finite) set than any which he has taken hitherto. Hilbert would treat of infinite sets by the same methods used for finite sets, as if he could comprehend them in their entirety. Gentzen refers to this point of view as the "as if" point of view. He presents various paradoxes which arise when the "as if" method is used without proper care.

This of course opens the question of what is "proper care." In the nature of things, the Brouwer method must fail to produce a paradox, since it never leaves the domain of the constructive finite. However the Brouwer method does not produce sufficient mathematical theory for physical and engineering uses. So Brouwer's method must be described as "excessive care."

A proposed way out of the difficulty is to base the "as if" method on an appropriate formal system, and use the Brouwer method to prove that the formal system is without a contradiction. For none of the various formal systems so far proposed has such a proof of freedom from contradiction been given. More serious still, a well known theorem of Gödel says that if a logic L_1 is used to prove the freedom from contradiction of a logic L_2, then L_1 must in some respects be stronger than L_2. So the above program will fall through unless one can point out some respect in which the Brouwer method is stronger than the "as if" method. Gentzen thinks he has found it.

His idea is to use the Brouwer method, involving the use of transfinite induction up to a certain ordinal α, to prove the freedom from contradiction of that part of the "as if" method which involves transfinite induction only up to an appropriate smaller ordinal β. If β is fairly large, the resulting "as if" method, though restricted, should be adequate for physics and engineering.

In the second book, Gentzen illustrates the above proposal by using the Brouwer method, with induction up to ϵ_0, to prove the freedom from contradiction of number theory with induction up to any ordinal less than ϵ_0. An important gap in the proof is the absence of a *constructive* proof that induction is valid up to ϵ_0. Gentzen himself comments on this gap, and expresses the belief that it will shortly be filled.

The present proof of freedom from contradiction is made considerably simpler than the earlier proof (in Mathematische Annalen—see above) by using Gentzen's LK-calculus, rather than his NK-calculus.