A THEOREM ON MATRICES OVER A COMMUTATIVE RING

NEAL H. McCOY

1. **Introduction.** Let R be an arbitrary commutative ring with unit element 1, and $R[\lambda]$ the ring of polynomials in the indeterminate λ, with coefficients in R. If A is a matrix of order n, with elements in R, the set of all elements $g(\lambda)$ of $R[\lambda]$, such that $g(A)=0$, is an ideal which we shall call the *minimum ideal* of A. The element $f(\lambda)=|\lambda-A|$ of $R[\lambda]$ is the *characteristic function* of A, and the principal ideal $(f(\lambda))$ may be called the *characteristic ideal* of A.* In a recent note,† it was shown that the minimum ideal of a matrix can be characterized in a manner generalizing Frobenius' characterization of the minimum function of a matrix for the case in which the coefficient domain is a field.‡ It was also shown that, in $R[\lambda]$, the prime ideal divisors of the minimum ideal coincide with those of the characteristic ideal. If R is specialized to be an algebraically closed field, this result yields the familiar theorem to the effect that the distinct linear factors of the characteristic function of A coincide with the distinct linear factors of the minimum function of A. It is the primary purpose of the present note to generalize, in a similar way, the well known theorem of Frobenius concerning the characteristic roots of a polynomial in two or more commutative matrices—or, more precisely, an extension of this theorem which we shall now describe in some detail.

Let K denote an algebraically closed field, and let us say that the matrices A_i, $(i=1, 2, \cdots, m)$, with elements in K, have property P, if the characteristic roots of every scalar polynomial $f(A_1, A_2, \cdots, A_m)$, with coefficients in K, are all of the form $f(\lambda_1, \lambda_2, \cdots, \lambda_m)$ where λ_i is a characteristic root of A_i, $(i=1, 2, \cdots, m)$.

In a previous paper,§ the following statements were shown to be equivalent:

I. The matrices A_i, $(i=1, 2, \cdots, m)$, have property P.

* The terms *minimum ideal* and *characteristic ideal* are used merely to emphasize that they generalize the usual notions of minimum and characteristic functions, respectively.

† Neal H. McCoy, *Concerning matrices with elements in a commutative ring*, this Bulletin, vol. 45 (1939), pp. 280–284.

‡ For the classical theorems concerning the characteristic and minimum functions and related topics, see C. C. MacDuffee, *The Theory of Matrices*, chap. 2, or J. H. M. Wedderburn, *Lectures on Matrices*, chap. 2.

§ N. H. McCoy, *On the characteristic roots of matric polynomials*, this Bulletin, vol. 42 (1936), pp. 592–600. Hereafter this will be referred to as M.