A NOTE ON THE WEIERSTRASS CONDITION IN THE CALCULUS OF VARIATIONS*

M. R. HESTENES AND W. T. REID

The present note is concerned with the proof of a simple property of the Weierstrass E-function which, so far as the authors know, has not been pointed out before. For the sake of generality, the result will be given for the general problem of the Lagrange or Bolza type. The result for such a problem in non-parametric form is given in Theorem I below, and the analogous result for the parametric problem is presented in Theorem II.

For the non-parametric problem let $f(x, y_1, \cdots, y_n, p_1, \cdots, p_n) = f(x, y, p)$ denote the integrand function and $\phi_\alpha(x, y, p)$, $(\alpha = 1, \cdots, m < n)$, the auxiliary expressions.† It will be assumed that the functions f, ϕ_α are continuous and have continuous derivatives of the first two orders in a region‡ $\mathcal{R}$ of (x, y, p)-space. By an *admissible set*

$$(x, y_1, \cdots, y_n, p_1, \cdots, p_n, \lambda_1, \cdots, \lambda_m) = (x, y, p, \lambda)$$

will be meant one such that (x, y, p) is in $\mathcal{R}$ and satisfies the equations $\phi_\alpha = 0$. Let $F(x, y, p, \lambda) = f(x, y, p) + \lambda_\alpha \phi_\alpha(x, y, p)$. Here and elsewhere in this note the tensor analysis summation convention is used.

THEOREM I. *Suppose N is a region in (x, y, p, λ)-space such that at each admissible set (x, y, p, λ) of N the inequality*

$$\begin{aligned} E(x, y, p, \lambda, q) \equiv F(x, y, q, \lambda) &- F(x, y, p, \lambda) \\ &- (q_i - p_i) F_{p_i}(x, y, p, \lambda) \geqq 0 \end{aligned} \tag{1}$$

holds for every set (q_i) for which (x, y, q, λ) is admissible. If the matrix

$$\left\| \begin{matrix} F_{p_i p_j} & \phi_{\beta p_i} \\ \phi_{\alpha p_j} & 0 \cdot \delta_{\alpha\beta} \end{matrix} \right\|, \qquad i, j = 1, \cdots, n; \alpha, \beta = 1, \cdots, m, \tag{2}$$

* Presented to the Society, December 30, 1938.

† For a more detailed formulation of the problems of Lagrange and Bolza the reader is referred, for example, to Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 673–742, Morse, *Sufficient conditions in the problem of Lagrange with variable end-conditions*, American Journal of Mathematics, vol. 53 (1931), pp. 517–546, or Bliss, *The Problem of Bolza in the Calculus of Variations*, mimeographed notes of lectures delivered winter, 1935, at the University of Chicago.

‡ By "region" we shall understand "open region." It is to be noted that in the following theorems no use is made of the region's being open with respect to the (x, y) or (y) variables. Consequently, the hypotheses of the theorems could be weakened in this respect.