THE STIELTJES MOMENT PROBLEM FOR FUNCTIONS OF BOUNDED VARIATION

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1. **Introduction.** We shall establish the following theorem, which at first sight appears quite unexpected:

THEOREM 1. *Any sequence* $\{\mu_n\}$ *of real numbers can be represented in the form*

$$(1.1)\qquad \mu_n = \int_0^\infty t^n d\alpha(t), \qquad n = 0, 1, 2, \cdots, \qquad \int_0^\infty |d\alpha(t)| < \infty.$$

The problem of determining necessary and sufficient conditions for a sequence of numbers $\{\mu_n\}$ to have the form

$$(1.2)\qquad \mu_n = \int_0^\infty t^n d\alpha(t), \qquad \alpha(t) \text{ non-decreasing, } n = 0, 1, 2, \cdots,$$

was set and solved by T. J. Stieltjes. It would be natural to attempt to generalize the problem by requiring merely that $\alpha(t)$ should be a function of bounded variation on $(0, \infty)$; but the generalized problem has, as Theorem 1 shows, a trivial solution.

To establish Theorem 1, we shall exhibit an arbitrary real sequence $\{\mu_n\}$ as the difference of two sequences $\{\lambda_n\}$ and $\{\nu_n\}$, each of the form (1.2).† The construction will also lead to the result that any sequence $\{\mu^n\}$ of positive numbers of sufficiently rapid growth has the form (1.2); it is sufficient, for example, that

$$(1.3)\qquad \mu_0 \geqq 1, \qquad \mu_n \geqq (n\mu_{n-1})^n, \qquad n \geqq 1.$$

A specimen sequence satisfying (1.3) is $\mu_0 = 1, \mu_n = n^{n^n}$, $(n = 1, 2, \cdots)$.

As an application‡ of Theorem 1, it will be shown that

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† Added in proof: Other proofs of Theorem 1 have been given by G. Pólya (*Sur l'indétermination d'un problème voisin du problème des moments*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp. 708–711). Pólya points out that a theorem of which Theorem 1 is an immediate consequence was proved by É. Borel in 1894.

‡ For another application of Theorem 1, see J. Shohat, *Sur les polynomes orthogonaux généralisés*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp 556–558.