A NOTE ON DIVISIBILITY SEQUENCES*

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1. **Introduction.** A sequence of rational integers

$$(u): \quad u_0, u_1, u_2, \cdots, u_n, \cdots$$

is called a *divisibility sequence* if u_r divides u_s whenever r divides s, and any integer M dividing terms of (u) with positive suffix is called a divisor of (u). The suffix s is called a *rank of apparition of* M if $u_s \equiv 0 \pmod{M}$, but $u_r \not\equiv 0 \pmod{M}$ if r is a proper divisor of s. It follows from a previous note of mine in this Bulletin (Ward [1]) that a necessary and sufficient condition that every divisor of (u) shall have only *one* rank of apparition is that (u) have the following property:

A. *If* $c=(a, b)$, *then* $u_c=(u_a, u_b)$ *for every pair of terms* u_a, u_b *of* (u).

Assume that no $u_r=0$, $(r>0)$. Then we may introduce numbers

$$[n, r] = u_n \cdot u_{n-1} \cdot \cdots \cdot u_{n-r+1}/u_1 \cdot u_2 \cdot \cdots \cdot u_r,$$
$$r = 1, \cdots, n; n = 1, 2, \cdots,$$

which we call the *binomial coefficients belonging to* (u).†

In a previous paper (Ward [1]), I proved a result equivalent to the following theorem:

THEOREM 1. *If every divisor of* (u) *has only one rank of apparition, the binomial coefficients belonging to* (u) *are rational integers.*

I give here a simple sufficient condition for integral binomial coefficients applicable when the divisors of (u) have several ranks of apparition.

2. **Main theorem.** Let (v) be any sequence of rational integers subject to the single condition $v_r \neq 0$, $(r>0)$. The sequence (u) will be said to have the property C if

$$u_n = \prod_{d|n} v_d,$$

the product being extended over all divisors d of n.

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† If $u_n=n$, they reduce to ordinary binomial coefficients. For their properties for general (u), see Ward [2].