ON THE DETERMINANT OF AN AUTOMORPH OF A NONSINGULAR SKEW-SYMMETRIC MATRIX

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Let G be the skew-symmetric matrix of order $2n$,

$$G = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix},$$

where E_n is the unit matrix of order n. If F is a matrix which satisfies

$$FGF' = G, \tag{1}$$

then $|F|^2 = 1$, so that $|F| = \pm 1$. That $|F| = +1$ is well known and is in fact a consequence of a theorem of Frobenius.* A simple proof communicated to me by Professor Wintner depends on the polar factorization of F, which reduces the problem at once to the case in which F is orthogonal. This proof is, of course, not valid in any field. It is our intention here to give a simple direct proof, applicable in any field, of the fact that $|F| = +1$.

On writing F as a matrix of matrices of order n, $F = (F_{ij})$, $(i, j = 1, 2)$, we have, as a consequence of (1),

$$\begin{aligned} F_{11}F'_{12} - F_{12}F'_{11} &= F_{21}F'_{22} - F_{22}F'_{21} = 0, \\ F_{11}F'_{22} - F_{12}F'_{21} &= F_{22}F'_{11} - F_{21}F'_{12} = E_n. \end{aligned} \tag{2}$$

Let $|F_{11}| \neq 0$. Then

$$F = \begin{pmatrix} F_{11} & F_{12}F'_{11} \\ F_{21} & F_{22}F'_{11} \end{pmatrix} \begin{pmatrix} E_n & 0 \\ 0 & (F'_{11})^{-1} \end{pmatrix}.$$

On, applying (2), we have

$$|F'_{11}|\,|F| = \begin{vmatrix} F_{11} & F_{12}F'_{11} \\ F_{21} & F_{22}F'_{11} \end{vmatrix} = \begin{vmatrix} F_{11} & F_{12}F'_{11} - F_{11}F'_{12} \\ F_{12} & F_{22}F'_{11} - F_{21}F'_{12} \end{vmatrix}$$

$$= \begin{vmatrix} F_{11} & 0 \\ F_{21} & E_n \end{vmatrix} = |F_{11}|.$$

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* G. Frobenius, *Ueber die schiefe Invariante einer bilinearen oder quadratischen Form*, Journal für die reine und angewandte Mathematik, vol. 86 (1879), pp. 44–71; in particular, p. 48. See A. Wintner, *On linear conservative dynamical systems*, Annali di Matematica Pura ed Applicata, (4), vol. 13 (1934–1935), pp. 105–112.