CONCERNING MATRICES WITH ELEMENTS IN A COMMUTATIVE RING*

NEAL H. MCCOY

1. **Introduction.** In a study of the algebraic properties of matrices with elements in a field, some of the most fundamental theorems are those having to do with the concepts of characteristic function and minimum function.† It is the purpose of the present note to suitably generalize some of the leading theorems in this connection to the case of matrices with elements in an arbitrary commutative ring R with unit element 1. For the most part, the theorems as well as the proofs are obtained by suitable generalizations of familiar theorems and proofs in the case in which the coefficient domain is restricted to be a field. However, the degree of generality here obtained seems to be of sufficient interest to warrant a brief account.

Let A denote a square matrix with elements in R. As will be indicated in §2, it is easy to define, in the usual way, the characteristic function $f(\lambda)$, and to show that $f(A)=0$. We shall call the principal ideal $(f(\lambda))$, in the ring $R[\lambda]$, the *characteristic ideal* of A.‡ The set of all elements $g(\lambda)$ of $R[\lambda]$ such that $g(A)=0$ is clearly an ideal in $R[\lambda]$ which may be called the *minimum ideal* of A. In general, this ideal will not be principal. The terms *characteristic ideal* and *minimum ideal* are used merely to emphasize the analogy with the characteristic and minimum functions of A in case the coefficient domain is a field. The actual determination of the minimum ideal is a fundamental problem, a solution of which is obtained in §3. The result, as stated in Theorem 3, is seen to be a generalization of the well known theorem of Frobenius concerning the minimum function. This theorem is the leading result of the present note.

One direction in which we propose to generalize certain familiar results will be sufficiently indicated by the remark that in place of irreducible factors of the characteristic (minimum) function of A we use the prime ideal divisors of the characteristic (minimum) ideal of A. For example, it is easy to show that the prime ideal divisors of the

* Presented to the Society, February 26, 1938.

† For known results concerning the characteristic and minimum functions, see J. H. M. Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, 1934, chap. 2, or C. C. MacDuffee, *The Theory of Matrices*, chap 2. The former will be referred to hereafter as W, the latter as M.

‡ For fundamental definitions and properties of ideals, see van der Waerden, *Moderne Algebra*, or Krull, *Idealtheorie*.