REPRESENTATION OF NUMBERS IN TERNARY QUADRATIC FORMS

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We employ integral quaternions $t=t_0+t_1i_1+t_2i_2+t_3i_3$, where the coordinates t_i range over rational integers, while the i_1, i_2, i_3, satisfy the multiplication table

$$i_1^2 = i_2^2 = -2, \quad i_3^2 = -3, \quad i_2i_3 = 2i_1 - i_2 = (\overline{i_3i_2}),$$

$$i_3i_1 = -i_1 + 2i_2 = (\overline{i_1i_3}), \qquad i_1i_2 = -1 + i_3 = (\overline{i_2i_1}),$$

and $\bar{t}=t_0-t_1i_1-t_2i_2-t_3i_3$ is the conjugate to t. The norm $N(t)$ of t is $t\bar{t}=\bar{t}t=t_0^2+2t_1^2+2t_2^2+2t_1t_2+3t_3^2$. The norm of a product of two quaternions equals the product of their norms, and $\overline{vt}=\bar{t}\bar{v}$ for any two quaternions. The associative law $rs\cdot t=r\cdot st$ holds.

The quaternary quadratic $Q=t_0^2+2t_1^2+2t_2^2+2t_1t_2+3t_3^2$ has determinant 9, the g.c.d. of the literal coefficients of the adjoint to Q is 3, and the second concomitant of Q represents no residues 1 modulo 3, and as there is only one form of determinant 9 with these properties in Charve's table* of reduced quaternary quadratic forms, Q belongs to a genus of one class. Since Q represents 1 for two values of $t_0, \cdots, t_3$, we have,† a proper quaternion being defined as one having coprime coordinates, the following theorem:

THEOREM 1. *A proper quaternion $v=v_0+v_1i_1+v_2i_2+v_3i_3$ whose norm is divisible by a positive integer m prime to 6 has exactly two right-divisors (left-divisors) t and $-t$ of norm m.*

Every proper pure quaternion $s=s_1i_1+s_2i_2+s_3i_3$ of norm km^2 is of form $\bar{t}at$ where $N(a)=k$ and $N(t)=m$. For, $s=vt$ where $N(t)=m$ by Theorem 1; $\bar{s}=-s=\bar{t}\bar{v}$, and $\bar{t}$ is a left-divisor of s. Hence, since $N(v)=km$, $\bar{t}$ is a left-divisor of the proper quaternion v, $v=\bar{t}a$. Hence $s=\bar{t}at$, $N(a)=k$. Clearly a is pure since $\bar{t}\bar{a}t=-\bar{t}at$, $\bar{a}=-a$.

THEOREM 2. *Consider the equation $24n+1=x_1^2+2x_2^2+2x_3^2-2x_2x_3$. If $24n+1=m^2$, $(m>0)$, then all proper solutions are of type A if $m\equiv 1$ (mod 4) but of type B if $m\equiv 3$ (mod 4), where*

$$A\colon\ x_1\equiv \pm 1 \pmod{12}, \qquad B\colon\ x_1\equiv \pm 5 \pmod{12}.$$

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* L. Charve, Comptes Rendus de l'Académie des Sciences, vol. 96 (1883), p. 773.

† G. Pall, *On the factorization of generalized quaternions*, submitted to the Duke Mathematical Journal.