THE RELATION OF PERFECT SETS OF MEASURE ZERO TO CERTAIN CLASSES OF FUNCTIONS

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1. **Introduction.** A real-valued function of a real variable defined on a set A is said to satisfy the (N) condition on A if the image set $f(E)$ of every set E of measure zero in A is of measure zero. Lusin has shown* that, when A is an interval and the function is continuous, the (N) condition is satisfied when this property holds on every perfect set of measure zero in the interval. In this paper we extend this result to a much wider class of functions and point out two consequences. The first concerns a generalization of a theorem due to Rademacher† which states that the (N) condition is necessary and sufficient in order that a continuous function transform measurable sets into measurable sets. The second gives a condition necessary and sufficient for the uniform convergence to an absolutely continuous function of certain sequences of absolutely continuous functions. In the last section a covering property of every perfect set of measure zero is pointed out.

We shall denote the Lebesgue measure of E by mE and its outer measure by $|E|$.

2. **The** (N) **condition.** We prove the following theorem:

THEOREM 1. *Let $f(x)$ be defined on A and satisfy the conditions:*

(1) *the image of any portion‡ of A with respect to $f(x)$ is measurable, and*

(2) *the set of solutions x of the equation $y=f(x)$ is a closed set with respect to A, for all values of y in $(-\infty, \infty)$ except at most for a set of measure zero.*

Let $E \subset A$ with $mE=0$ and $P \subset A$, $mP=0$, and P perfect with respect to A. Then $mf(P)=0$ for all P implies $mf(E)=0$.

PROOF. Assuming that a set E exists for which $m(E)=0$, and $|f(E)|=k>0$, we let E be covered by a set of intervals B_1 with $mB_1<1/2$. Let C_1 be a finite number of these intervals for which $|f(C_1\cdot E)|>k/2$. In a similar manner cover $C_1\cdot E$ by B_2 with $B_2\subset C_1$ and $mB_2<1/4$, and let C_2 be a finite number of the intervals of B_2 for which $|f(C_2\cdot E)|>k/2$. Continuing this process, obtain $C_i\subset C_{i-1}$

* N. Lusin, *Intégrale et Série Trigonométrique*, Moscow, 1915, p. 109 (in Russian).

† H. Rademacher, Monatshefte für Mathematik und Physik, vol. 27 (1916), pp. 183–291.

‡ A portion of a set is the intersection of the set and an interval.