THE EQUIVALENCE OF SEQUENCE INTEGRALS AND NON-ABSOLUTELY CONVERGENT INTEGRALS*

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This note completes and extends some results previously obtained.† Let the function $f(x)$ be measurable, and finite almost everywhere on (a, b). Let $s_n(x)$ be a sequence of summable functions such that $s_n=f$ on a set E_n, $s_n=0$ elsewhere, $E_n \supset E_{n-1}$, and $mE_n \to b-a$. If $\int_a^x s_n dx$ tends to a continuous function $\phi(x)$, then f is, by definition, *totally integrable in the sequence sense to* $\phi(x)=TS(f, a, x)$. It has been shown that if $f(x)$ is integrable in the generalized Denjoy sense to $F(x)=\int_a^x f(x)dx$, then there exists $TS(f, a, x)=F(x)$.‡ Such a function $TS(f, a, x)$ is generalized absolutely continuous (ACG),§ since $F(x)$ is (ACG). A function $TS(f, a, x)$ was constructed‖ which was not (ACG) and consequently not equal to $F(x)$. This raised the question as to whether or not the property of being (ACG) was sufficient to insure that $TS(f, a, x)=F(x)$. In the present note this question is answered in the negative, and necessary and sufficient conditions are determined for the relation $TS(f, a, x)=F(x)$.

We first construct a function $f(x)$ which is not summable, but which is integrable in a non-absolutely convergent sense, and then construct $TS(f, a, x)$ which is (ACG) and not equal to $F(x)=\int_a^x f dx$. Let G be a Cantor set on (a, b) with $mG>0$, and let (α_i, β_i) be the intervals complementary to G. On (α_i, β_i) construct f_i such that $\int_{\alpha_i}^{\beta_i} f_i dx$ exists as a non-absolutely convergent integral with β_i the single point of non-summability of f_i, with $\int_{\alpha_i}^{\beta_i} f_i dx=0$, and with $\left|\int_{\alpha_i}^{x} f_i dx\right| < \beta_i-\alpha_i$ for x on (α_i, β_i). Let $f(x)=f_i(x)$ on (α_i, β_i), and $f(x)=0$ elsewhere. Then $F(x)=\int_a^x f dx$ exists as a non-absolutely convergent integral, and $F(x)=0$ for x a point of G. Consider the set of intervals (α_i, β_i) ordered in any way. Then take the first n intervals of this set and order them from left to right into the set $(\alpha_1', \beta_1'), \cdots, (\alpha_n', \beta_n')$. To the right of each interval (α_j', β_j') there is an interval $\lambda_{ni}=(\beta_j', \alpha_{j+1}')$, where i is the subscript that (α_i, β_i) has in the original ordering

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† Transactions of this Society, vol. 41 (1935), pp. 171–192. In what follows this paper will be referred to as T.

‡ T, p. 186, Theorem 6.

§ Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 152, §9.

‖ T, pp. 189–191.